MA372: Differential Equations

Dylan C. Beck

Acknowledgements

Primarily, the contents of this document were created in the Spring 2023 and Fall 2023 semesters at Baker University. I am grateful to the students in MA372 (Differential Equations) — especially those who assisted in the enhancement of these notes with comments and suggestions.

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Chapter 0

Differential and Integral Calculus

Broadly speaking, differential calculus is the study of instantaneous change. Early on in a first calculus course, students learn that the derivative of a function at a point measures the slope of the line tangent at that point; the slope of the tangent line at a point is simply limit of the slopes of the secant lines passing through the specified point, and these slopes measure the average rate of change of the function. Consequently, the derivative measures the instantaneous change of a function. Bearing this in mind, calculus is immediately applicable in a wide range of fields — from physics and engineering to biology, chemistry, and medicine. Conversely, it is the aim of integral calculus to quantify change over time given the instantaneous rate of change. Combined, differential and integral calculus constitute an indispensable tool in many applied sciences today.

0.1 Limits and Continuity

Calculus is the study of change in functions. Essentially, a **function** is simply a rule that assigns to each input x one and only one output $y = f(x)$. Often, in this course, we will simply consider **real functions**, i.e., functions that are defined such that their inputs and outputs are **real numbers**. We are unwittingly very familiar with real numbers: the real numbers $\mathbb{R}$ include zero, all positive and negative whole numbers, all positive and negative rational numbers (or fractions), all positive and negative square roots of positive rational numbers, and transcendental numbers like π and e.

We will use the notation $f : \mathbb{R} \to \mathbb{R}$ to express that f is a function whose **domain** is the real numbers $\mathbb{R}$ and whose **codomain** is the real numbers $\mathbb{R}$. Explicitly, the domain of a function is the set of all possible inputs of a function, and the codomain of a function is the set of all possible outputs of the function. Even more, the collection of all possible outputs of a function is the **range** of the function. We will adopt the **set-builder** notation for the domain and range of a function f.

$D_f = \{x \in \mathbb{R} \mid f(x) \text{ is a real number}\}$ consists of real numbers x such that $f(x)$ is a real number.

$R_f = \{f(x) \in \mathbb{R} \mid x \in D_f\}$ consists of real numbers $f(x)$ such that x lies in the domain of f.

Example 0.1.1. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. By definition, this function outputs the real number x that is input. We refer to this as the **identity function** on the real numbers. Consequently, the domain of f is $D_f = \mathbb{R}$ because the output of any real number is a real number, and the range of f is $R_f = \mathbb{R}$ because every real number is the output of itself.

Caution: the domain of a real function might not be all real numbers; the range of a real function might not be all real numbers, either, as our next pair of examples illustrate.

Example 0.1.2. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. By definition, this function outputs the square x^2 of the real number x that is input. Certainly, the square of any real number is a real number, hence the domain of f is $D_f = \mathbb{R}$; on the other hand, the only real numbers that are the square of another real number are the non-negative real numbers. Explicitly, for any real number x, the real number $f(x) = x^2$ is a non-negative real number, i.e., we have that $x^2 \geq 0$. Consequently, the codomain of f is $\mathbb{R}$, but the range of f is $R_f = \mathbb{R}_{\geq 0} = \{y \in \mathbb{R} \mid y \geq 0\}$.

Example 0.1.3. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. By definition, this function outputs the square root $\sqrt{x}$ of the real number x that is input. We cannot take the square root of a negative real number, hence the domain of f consists of all non-negative real numbers, i.e., we have that $D_f = \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$; on the other hand, every non-negative real number can be realized as the square root of a non-negative real number. Explicitly, for any non-negative real number y, the real number y^2 satisfies that $y = \sqrt{y^2} = f(y^2)$. Consequently, the codomain of f is $\mathbb{R}$, but once again, the range of f is $R_f = \mathbb{R}_{\geq 0} = \{y \in \mathbb{R} \mid y \geq 0\}$.

Generally, the restrictions on the domain of a real function consist of the following situations.

(a.) We cannot divide by zero.

(b.) We cannot take the even root of a negative real number.

(c.) We cannot take the logarithm of a non-positive real number.

Occasionally, it is necessary to split the domain or the range of a function into distinct chunks of the real number line. By the above rule, the domain of the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{-1}$ consists of all nonzero real numbers. Consequently, we can certainly realize the domain of f as $D_f = \{x \in \mathbb{R} \mid x \neq 0\}$, but it is sometimes more convenient to describe this set using the **union** symbol $\cup$. Put simply, the union symbol $\cup$ functions as the logical connective "or." Clearly, a nonzero real number is either positive or negative, hence we can partition the domain of f into those real numbers that are positive and those real numbers that are negative. We achieve this with the union symbol as $D_f = \{x \in \mathbb{R} \mid x > 0\} \cup \{x \in \mathbb{R} \mid x < 0\}$. Even more, we learn in college algebra (or earlier) that the set of real numbers x satisfying the **inequalities** $x > 0$ and $x < 0$ can be described respectively using the **open intervals** $(0, \infty)$ and $(-\infty, 0)$. Consequently, in **interval notation**, the domain of the real function $f(x) = x^{-1}$ is given by $D_f = (-\infty, 0) \cup (0, \infty)$.

Exercise 0.1.4. Compute the domain and range of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

Exercise 0.1.5. Compute the domain and range of the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^{-3}$.

Exercise 0.1.6. Compute the domain and range of the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \frac{1}{\ln(x)}$.

Consider a function $f : \mathbb{R} \to \mathbb{R}$ whose domain is D_f. Given any real number a in D_f, we say that the **limit** of $f(x)$ as x approaches a is the quantity L (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$. Put another way, the quantity L can be made arbitrarily close to the value of $f(x)$ by taking x to be sufficiently close in value to a. Conveniently, if the quantity L exists, then we write $L = \lim_{x \to a} f(x)$.

Example 0.1.7. Let us compute the limit of $f(x) = x^2$ as x approaches $a = 1$ using the definition. Computing the limit is essentially like playing a game of limbo: we are handed a real number $\varepsilon > 0$ (the limbo bar), and our challenge is to find a real number $\delta > 0$ such that $|x^2 - 1| < \varepsilon$ whenever we assume that $|x - 1| < \delta$. Of course, we are at liberty to take δ as small as necessary to ensure that $|x^2 - 1| < \varepsilon$. We may therefore assume that $0 < \delta \leq 1$. Considering that $x^2 - 1 = (x - 1)(x + 1)$, if we assume that $|x - 1| < \delta \leq 1$, then we must have that $0 < x < 2$, from which it follows that $|x + 1| \leq |x| + 1 = x + 1 < 3$ by the Triangle Inequality. Consequently, we have that

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x - 1||x + 1| < 3\delta.$$

Last, if we wish to have that $|x^2 - 1| < \varepsilon$, then we should choose δ to be the minimum of 1 and $\frac{\varepsilon}{3}$.

One-sided limits can be defined analogously to the limit above: the **left-hand limit** of $f(x)$ as x approaches a is the quantity L^- (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $-\delta < x - a < 0$ implies that $|f(x) - L^-| < \varepsilon$. Likewise, the **right-hand limit** of $f(x)$ as x approaches a is the quantity L^+ (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $0 < x - a < \delta$ implies that $|f(x) - L^+| < \varepsilon$.

$L^- = \lim_{x \to a^-} f(x)$ is the symbolic way to express the left-hand limit of $f(x)$ as x approaches a.

$L^+ = \lim_{x \to a^+} f(x)$ is the symbolic way to express the right-hand limit of $f(x)$ as x approaches a.

Ultimately, the two-sided limit exists if and only if the left- and right-hand limits exist and are equal; thus, the two-sided limit is equal to the common value of the left- and right-hand limits.

$$L^- = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = L^+$$

Graphically, it is possible to compute the two-sided limit L of some functions $f(x)$ as x approaches a by tracing one's finger along the graph of $f(x)$ from the left- and right-hand sides.

Example 0.1.8. Let us graphically compute the limit of $f(x) = x^2$ as x approaches $a = 1$. Using the graph of $f(x) = x^2$, we find that the limit is 1. Particularly, if we trace the graph with our left pointer finger, moving from left to right toward the point $x = 1$, our finger stops at $y = f(1) = 1$. Likewise, if we trace the graph with our right pointer finger moving from right to left toward $x = 1$, our finger stops at $y = f(1) = 1$. Put in the language of calculus, we have that $L^- = 1 = L^+$.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at a real number a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Explicitly, we require three things to be true of the function $f(x)$ in this case.

1.) We must have that f is defined at the real number a, i.e., $f(a)$ must be in the range of f.

2.) We must have that $\lim_{x \to a^-} f(x) = f(a)$, i.e., the left-hand limit of f at a must be $f(a)$.

3.) We must have that $\lim_{x \to a^+} f(x) = f(a)$, i.e., the right-hand limit of f at a must be $f(a)$.

Consequently, if any of these criteria is violated, then the function f cannot be continuous at a.

Example 0.1.9. One of the easiest ways to detect that a function is not continuous at a real number a is to observe that the function is not defined at a. Explicitly, the function $f(x) = \frac{1}{x}$ is not continuous at $a = 0$ because the domain of f excludes $a = 0$ (since we cannot divide by zero).

Example 0.1.10. Consider the function $f : \mathbb{R} \to \mathbb{R}$ that is defined **piecewise** as follows.

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and} \\ -1 & \text{if } x < 0 \end{cases}$$

Graphically, if we trace our fingers along f from the left-hand side, when we arrive at $a = 0$ from the left-hand side, we find that the limiting value here is -1; however, if we trace our fingers along f from the right-hand side, when we arrive at $a = 0$ from the right-hand side, we find that the limiting value here is 1. Consequently, the function $f(x)$ is not continuous at $a = 0$.

Example 0.1.11. Let us prove by definition that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is continuous for all real numbers a. Observe that f is defined piecewise as follows.

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \text{ and} \\ -x & \text{if } x < 0 \end{cases}$$

Consequently, it suffices to show that $g(x) = x$ and $h(x) = -x$ are everywhere continuous. Given real numbers $\varepsilon_1, \varepsilon_2 > 0$, we must find real numbers $\delta_1, \delta_2 > 0$ such that $|x - a| < \varepsilon_1$ whenever $|x - a| < \delta_1$ and $|-x - (-a)| < \varepsilon_2$ whenever $|x - a| < \delta_2$. Considering that the absolute value is multiplicative, we have that $|-x - (-a)| = |-x + a| = |-(x - a)| = |x - a|$, we may simply take the real numbers $\delta_1 = \varepsilon_1$ and $\delta_2 = \varepsilon_2$. We conclude that $g(x) = x$ and $h(x) = -x$ are continuous for all real numbers a so that $f(x) = |x|$ is continuous for all nonzero real numbers by the piecewise definition of $f(x)$ prescribed above. We are done as soon as we show that

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x|.$$

By continuity of the functions $g(x)$ and $h(x)$ and by definition of $|x|$, the left-hand limit is given by $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} h(x) = h(0) = 0$, and the right-hand limit is $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} g(x) = g(0) = 0$.

Generally, continuity can be defined as a property of a function on any **subset** of its domain, i.e., on any collection of real numbers that lie in the domain. Often, we will consider functions that are continuous on their entire domain, but it is possible that a function is not continuous at some point in its domain. We say that a function f is **discontinuous** at a real number a if f is not continuous at the real number a. By the above three criteria, we can classify these **discontinuities**.

- We say that f has a **removable discontinuity** at a real number a if a is not in the domain of f but the left- and right-hand limits of f at a exist and are equal, i.e., $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.
- We say that f has a **jump discontinuity** at a real number a if both of the left- and right-hand limits of f at a exist but are not equal, i.e., $\lim_{x \to a^-} f(x) = L^- \neq L^+ = \lim_{x \to a^+} f(x)$.
- We say that f has an **essential discontinuity** at a real number a if either the left- or the right-hand limit of f at a does not exist, i.e., either $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ does not exist.

Often, if a function f is continuous for every real number in its domain D_f, we will say that the function is **continuous** on its domain. Explicitly, if the domain of a function f is all real numbers and f is continuous on its domain, then we will say that f is **everywhere continuous**. Graphically, we may detect that a function is continuous if we can draw it without lifting our pencil.

Example 0.1.12. We can graph $|x|$ without lifting our pencil, hence it is everywhere continuous.

Example 0.1.13. We cannot graph x^{-2} without lifting our pencil at $x = 0$, hence x^{-2} is not continuous at $a = 0$. On the other hand, for all real numbers a other than $a = 0$, we can graph this function without lifting our pencil, hence x^{-2} is continuous on its domain $(-\infty, 0) \cup (0, \infty)$.

Continuous functions abound: **polynomial** functions such as $x^3 - 2x^2 + x - 7$ and **exponential** functions such as e^x are defined for all real numbers and are everywhere continuous. Likewise, the **trigonometric** functions $\sin(x)$ and $\cos(x)$ are defined for all real numbers and are everywhere continuous. **Logarithmic** functions such as $\ln(x)$ and $\log(x)$ and **algebraic** functions such as $\sqrt{x}$ and $x^{3/2}$ are defined for all positive real numbers and are continuous on their domains. Further, addition, subtraction, multiplication, division, composition, and any finite combination of these operations on continuous functions result in functions that are typically continuous on their domains.

0.2 Differentiation and L'Hôpital's Rule

Given any real numbers a and $h > 0$ and any real function $f(x)$ such that $f(a)$ and $f(a+h)$ are defined, consider the closed interval $[a, a+h]$ consisting of all real numbers x with $a \leq x \leq a+h$. We define the **secant line** of $f(x)$ over this interval as the line passing through the points $(a, f(a))$ and $(a+h, f(a+h))$. Observe that the slope of the secant line is given by the **difference quotient**

$$Q_a(h) = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

By taking the limit of $Q_a(h)$ as h approaches 0, we obtain the **derivative** of $f(x)$ at a

$$f'(a) = \lim_{h \to 0} Q_a(h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Of course, this limit might not exist; however, if it does, we interpret it geometrically as the slope of the line tangent to $f(x)$ at the point $(a, f(a))$. Given that the quantity $f'(a)$ exists, we say that $f(x)$ is **differentiable** at a. One fundamental interpretation of the derivative in the context of a function that measures something physical (e.g., velocity) is as the instantaneous rate of change.

Exercise 0.2.1. Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = x^3$.

Exercise 0.2.2. Use the limit definition of the derivative to compute $g'(x)$ for $g(x) = \frac{1}{x}$.

Exercise 0.2.3. Use the limit definition of the derivative to compute $h'(x)$ for $h(x) = \sqrt{x}$.

One of the most important properties of differentiable real functions is the following.

Proposition 0.2.4. *If a real function f is differentiable at a real number a, then f is continuous at a. Explicitly, a function that is differentiable at a point in its domain is necessarily continuous there. Conversely, there exists a function that is continuous but not differentiable on its domain.*

Proof. We will assume that f is differentiable at a real number a. Consequently, the limit

$$f'(a) = \lim_{h \to 0} Q_a(h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Using the substitution $x = a + h$, we have that $h = x - a$. Crucially, under this substitution, the limit of any function $g(h)$ as h approaches 0 is equal to the limit of the function $g(x - a)$ as x approaches a. (Verify this by definition of the limit.) Consequently, the following identity holds.

$$f'(a) = \lim_{x \to a} Q_a(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Considering that $x - a$ is a polynomial function, it is continuous at a, and we conclude that

$$\lim_{x \to a}(x - a) = a - a = 0.$$

Using the fact that the limit of a product is the product of limits (when both limits exist),

$$0 = f'(a) \cdot \lim_{x \to a}(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot x - a = \lim_{x \to a}[f(x) - f(a)]$$

yields the result that $\lim_{x \to a} f(x) = \lim_{x \to a}[f(a) + f(x) - f(a)] = \lim_{x \to a} f(a) + \lim_{x \to a}[f(x) - f(a)] = f(a)$.

Conversely, the function $|x|$ is continuous on its domain, but it is not differentiable at $a = 0$: indeed, by Example 0.1.10, the piecewise function $f(x)$ satisfying that $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$ is not continuous because the left- and right-hand limits do not agree at 0. One can readily verify that this function is exactly the derivative of $|x|$, hence the claim holds. □

Computing limits by definition is even more tedious than it looks, but luckily, there are plenty of tools that allow us to compute derivatives of functions without ever touching a limit. Particularly,

- the **Power Rule** says that if $f(x) = x^r$ for some real number r, then $f'(x) = rx^{r-1}$;
- the **Product Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x);$$

- the **Quotient Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}; \text{ and}$$

- the **Chain Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f \circ g(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) = [f' \circ g(x)] \cdot g'(x).$$

Computing the limit of a function that is continuous is quite easy: we may simply "plug and chug;" however, there exist functions that are not continuous. Even worse, when evaluating limits, we can encounter situations that result in an **indeterminate form** when the limit is the form

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

Theorem 0.2.5 (L'Hôpital's Rule)**.** *Given any real functions $f(x)$ and $g(x)$ that are differentiable for all real numbers x such that $a < x < b$ (with the possible exception of one point $x = c$ for some real number $a \leq c \leq b$), consider the following conditions.*

(1.) *We have that* $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ *or* $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\infty$.

(2.) *We have that* $g'(x) \neq 0$ *for any real number x such that $a < x < b$ and $x \neq c$.*

(3.) *We have that* $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ *exists.*

Granted that each of the above conditions holds, it follows that $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Exercise 0.2.6. Compute the limit of $f(x) = \dfrac{\ln(x)}{x^3 - 1}$ as x approaches $a = 1$.

Exercise 0.2.7. Compute the limit of $g(x) = (2x - \pi)\sec(x)$ as x approaches $a = \frac{\pi}{2}$ from the left.

Exercise 0.2.8. Compute the limit of $h(x) = \dfrac{\sin(x)}{\sin(x) + \tan(x)}$ as x approaches $a = 0$.

Exercise 0.2.9. If $\frac{d}{dx}\sin(x) = \cos(x)$, compute the limit of $f(x) = \dfrac{\sin(x)}{x}$ as x approaches $a = 0$.

Caution: Unfortunately, the above example is not a valid proof of this limit identity: in fact, this limit identity is needed to prove that $\frac{d}{dx}\sin(x) = \cos(x)$, so in order to prove this identity in a rigorous and non-circular manner, we must use tools from trigonometry and the Squeeze Theorem.

0.3 Implicit Differentiation

Curves in the Cartesian plane can be represented by an equation involving a function of two variables. Explicitly, we are familiar with such curves as $xy = 1$ and $y - x^2 = 0$; they are respectively the functions $y = f(x) = x^{-1}$ and $y = g(x) = x^2$. We refer to the functions $f(x)$ and $g(x)$ as the **explicit** forms of the curves. Unfortunately, it is not possible to write every curve in the Cartesian plane as a function of one variable: curves such as the unit circle $x^2 + y^2 = 1$ or the hyperbola $y^2 - x = 0$ cannot be represented as functions because they fail the **Vertical Line Test**; however, we will see throughout this semester that these curves provide important models in calculus. Curves that do not admit closed-form expressions of the form $y = f(x)$ can be written **implicitly**.

Under certain conditions, it is possible to find a "small enough" region in the Cartesian plane in which an implicit curve can be represented by a function; thus, in this "window," the slope and tangent line of such curves are well-defined. Consequently, we may define the **implicit derivative** by assuming that y is a function of x (on some "small window" in the plane) with derivative $y' = \frac{dy}{dx}$.

Example 0.3.1. Compute $\frac{dy}{dx}$ for the unit circle $x^2 + y^2 = 1$.

Solution. Considering the variable y as some function $y = f(x)$ of x and using the convention that $y' = \frac{dy}{dx}$, we may invoke the Chain Rule in order to determine that

$$0 = \frac{d}{dx}1 = \frac{d}{dx}(x^2 + y^2) = 2x + 2yy'.$$

Crucially, each time the derivative operator $\frac{d}{dx}$ encounters the variable y, we differentiate y as we would the function $y = f(x)$ that represents y locally. Consequently, if y is nonzero, then

$$\frac{dy}{dx} = y' = -\frac{2x}{2y} = -\frac{x}{y}.$$

Otherwise, the tangent line does not exist if $y = 0$ because $2x + 2yy' = 0$ has no solution if $y = 0$. ◇

Example 0.3.2. Compute $\frac{dy}{dx}$ for the parabola $y^2 - x = 0$.

Solution. By the Chain Rule applied to $y = f(x)$, we have that

$$0 = \frac{d}{dx}0 = \frac{d}{dx}(y^2 - x) = 2yy' - 1$$

so that $\frac{dy}{dx} = y' = (2y)^{-1}$ for all points (x, y) on the hyperbola such that y is nonzero. ◇

0.4 Exponential and Logarithmic Functions

Given any positive real number a, the **exponential** function with **base** a is given by $\exp_a(x) = a^x$. Crucially, the most important exponential function is simply $\exp(x) = e^x$: here, the base is **Euler's number** $e \approx 2.72$. Later, we will concern ourselves with the definition of Euler's number; for now, we need only recall the following properties of exponential functions for any real numbers x and y.

1.) $a^{x+y} = a^x a^y$

2.) $a^{x-y} = a^x a^{-y}$

3.) $a^{xy} = (a^x)^y$

4.) $(ab)^x = a^x b^x$ for any real number $b > 0$

We do not yet have the machinery available to use to prove the following, but it is true that

$$\frac{d}{dx}e^x = e^x.$$

Considering that $e^x > 0$ for all real numbers x, it follows that e^x is a strictly increasing function, hence it passes the **Horizontal Line Test** and must therefore admit an **inverse** function; we refer to this function as the **natural logarithmic** function $\ln(x)$. Put another way, we have that

$$e^{\ln(x)} = x \text{ for all real numbers } x > 0 \text{ and } \ln(e^x) = x \text{ for all real numbers } x.$$

Observe that the range of e^x is $(0, \infty)$, hence the domain of $\ln(x)$ is $(0, \infty)$. Conversely, the domain of e^x is $(-\infty, \infty)$, hence the range of $\ln(x)$ is $(-\infty, \infty)$. We will also simply assert that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

We may also deduce the following properties of logarithmic functions for any real numbers $x, y > 0$.

1.) $\log_a(xy) = \log_a(x) + \log_a(y)$

2.) $\log_a(x^r) = r\log_a(x)$ for all real numbers r

3.) $\log_a(xy^{-1}) = \log_a(x) - \log_a(y)$

4.) $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

Even more, for any real number $a > 0$, the exponential function $\exp_a(x) = a^x$ is differentiable for all real numbers x. Further, observe that $y = a^x$ is strictly positive for all real numbers x, hence the function $\ln(y) = x\ln(a)$ is well-defined. Using the Chain Rule, we find that

$$\frac{1}{y}\cdot y' = \frac{d}{dx}\ln(y) = \frac{d}{dx}[x\ln(a)] = \ln(a)\cdot\frac{d}{dx}x = \ln(a) \text{ and } \frac{d}{dx}a^x = \frac{d}{dx}y = \frac{dy}{dx} = y' = y\ln(a) = a^x\ln(a).$$

By a similar rationale as before, one can define the **logarithmic** function $\log_a(x)$ **base** a for any positive real number a as the function inverse of a^x; its domain is $(0, \infty)$, and its range is $(-\infty, \infty)$.

Exercise 0.4.1. Compute the derivative of $y = \log_a(x)$ by using the fact that $a^y = x$.

0.5 Inverse Trigonometric Functions

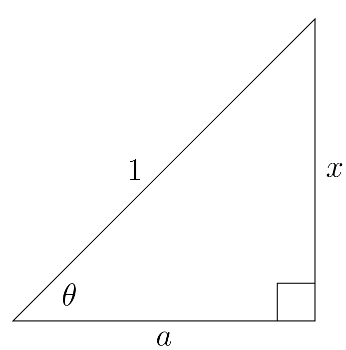
Even though the trigonometric functions like $\sin(x)$, $\cos(x)$, and $\tan(x)$ are **periodic**, we can find a region on the x-axis in which these functions pass the Horizontal Line Test and admit function inverses. Explicitly, the inverse trigonometric functions are denoted as follows.

$\arcsin(x) = \sin^{-1}(x)$	domain: $[-1, 1]$	range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\arccos(x) = \cos^{-1}(x)$	domain: $[-1, 1]$	range: $[0, \pi]$
$\arctan(x) = \tan^{-1}(x)$	domain: $(-\infty, \infty)$	range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

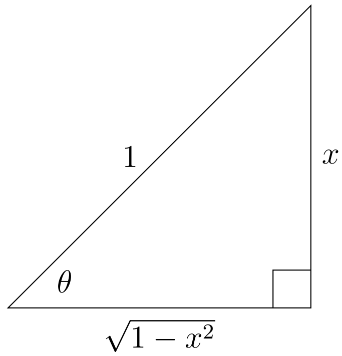
Considering that the input of the sine function is an angle, the output of the arcsine function is an angle. Consequently, if $x = \sin(\theta)$, then it follows by definition that $\theta = \arcsin(x)$ so that

$$\frac{d}{dx}\arcsin(x) = \frac{d\theta}{dx}.$$

Observe that $\sin(\theta)$ is the ratio of the opposite side and the hypotenuse of a right triangle, so we may construct a right triangle whose opposite side has length x and whose hypotenuse has length 1 in order to obtain $\sin(\theta) = x$. Our right triangle therefore has the following form.



By the **Pythagorean Theorem**, we must have that $x^2 + a^2 = 1$ so that $a = \sqrt{1 - x^2}$.

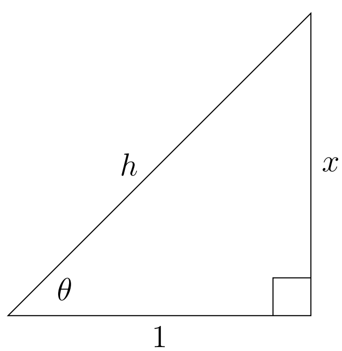


Using the Chain Rule, we can compute $\frac{d\theta}{dx}$. Explicitly, we have that

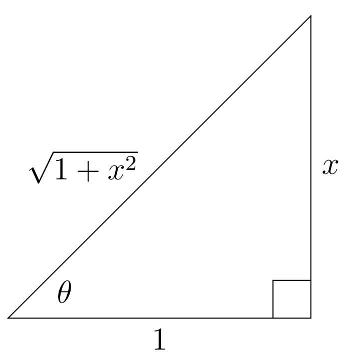
$$\cos(\theta) \cdot \frac{d\theta}{dx} = \frac{d}{dx}\sin(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\arcsin(x) = \frac{d\theta}{dx} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x^2}}.$$

Exercise 0.5.1. Use a right triangle involving 1, x, and $\sqrt{1-x^2}$ to compute $\frac{d}{dx}\arccos(x)$.

Using a similar idea as the one we employed to compute the derivative of $\arcsin(x)$ and $\arccos(x)$, we will set up a triangle with $\tan(\theta) = x$. Observe that $\tan(\theta)$ is the ratio of the opposite side and the adjacent side of a right triangle, so we may construct a right triangle whose opposite side has length x and whose adjacent side has length 1 in order to obtain $\tan(\theta) = x$.



Once again, by the Pythagorean Theorem, we find that $h^2 = x^2 + 1^2$ so that $h = \sqrt{1+x^2}$.

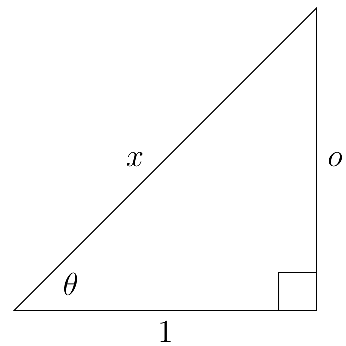


Using the Chain Rule, we can compute $\frac{d\theta}{dx}$. Explicitly, we have that

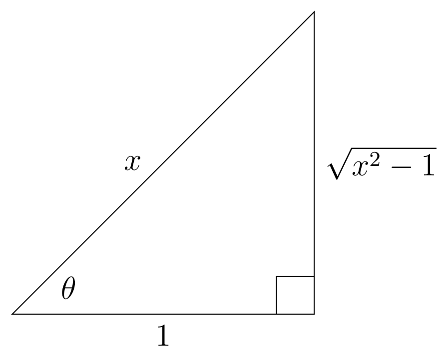
$$\sec^2(\theta) \cdot \frac{d\theta}{dx} = \frac{d}{dx}\tan(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\arctan(x) = \frac{d\theta}{dx} = \cos^2(\theta) = \frac{1}{1+x^2}.$$

Exercise 0.5.2. Use a right triangle involving 1, x, and $\sqrt{1+x^2}$ to compute $\frac{d}{dx}\operatorname{arccot}(x)$.

Last but not least, we will set up a triangle with $\sec(\theta) = x$. Observe that $\sec(\theta)$ is the ratio of the hypotenuse to the adjacent side of a right triangle, so we obtain the following diagram.



Once again, by the Pythagorean Theorem, we find that $x^2 = o^2 + 1^2$ so that $o = \sqrt{x^2 - 1}$.



Using the Chain Rule, we can compute $\dfrac{d\theta}{dx}$. Explicitly, we have that

$$\sec(\theta)\tan(\theta) \cdot \frac{d\theta}{dx} = \frac{d}{dx}\sec(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\operatorname{arcsec}(x) = \frac{d\theta}{dx} = \cos(\theta)\cot(\theta) = \frac{1}{x\sqrt{x^2-1}}.$$

Exercise 0.5.3. Use a right triangle involving 1, x, and $\sqrt{x^2-1}$ to compute $\frac{d}{dx}\operatorname{arccsc}(x)$.

0.6 Antidifferentiation

Considering that a derivative is a rate of change, it is natural in the applied sciences to begin with a rate of change and use it to estimate the net change of a process over time. Explicitly, if we observe that the velocity of a body is given by a function $f(x)$ over some interval of time, then we may seek a function $F(x)$ such that $F'(x) = f(x)$ over this interval of time. Given that such a function $F(x)$ exists and satisfies that $F'(x) = f(x)$, we refer to $F(x)$ as an **antiderivative** of $f(x)$.

Exercise 0.6.1. Prove that the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.

Exercise 0.6.2. Prove that the function $G(x) = x\ln(x) - x$ is an antiderivative of $g(x) = \ln(x)$.

Exercise 0.6.3. Prove that the function $H(x) = xe^x - e^x$ is an antiderivative of $h(x) = xe^x$.

Observe that for any antiderivative $F(x)$ of a function $f(x)$, there exists a family of antiderivatives indexed by the real numbers. Particularly, the function $G(x) = F(x) + C$ is an antiderivative of $f(x)$ for every real number C. Even more, by the **Mean Value Theorem**, every antiderivative of $f(x)$ is of the form $F(x) + C$ for some antiderivative $F(x)$ of $f(x)$ and some real number C. Consequently, we may define the **general antiderivative** or **indefinite integral** of $f(x)$ to be

$$\int f(x)\,dx = F(x) + C$$

for any real number C. By the familiar derivative rules, we obtain

- the **Power Rule**, i.e., $\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C$ for all real numbers $r \neq -1$ and
- the **Chain Rule**, i.e., $\int f'(g(x))g'(x)\,dx = f(g(x)) + C$.

Further, indefinite integration is **linear**: for all real functions $f(x)$ and $g(x)$, we have

- the **Multiples Rule** $\int kf(x)\,dx = k\big(\int f(x)\,dx\big)$ for all real numbers k and
- the **Sum Rule** $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Exercise 0.6.4. Compute the indefinite integral of $f(x) = x^{-1}$.

Exercise 0.6.5. Compute the indefinite integral of $g(x) = 2xe^{x^2}$.

Exercise 0.6.6. Compute the indefinite integral of $h(x) = 2\sin(x)\cos(x)$.

Circling back to the opening remarks of this section, we will assume that the velocity of a body over an interval of time is a continuous function $v(t)$. Even more, suppose that we note the position $s(t)$ of the particle at time $t = 0$, i.e., the quantity $s(0)$ is known. Considering that $s'(t) = v(t)$, it follows that $s(t)$ must differ from $\int v(t)\,dt$ by a constant C that depends on the quantity $s(0)$. We refer to this scenario as an **initial value problem** of the **differential equation** $s'(t) = v(t)$.

Example 0.6.7. Consider the velocity function $v(t) = 3t^2 - 4t + 2$ of a body whose position $s(t)$ at time $t = 0$ is given by $s(0) = 7$. Give an explicit formula for $s(t)$.

Solution. Observe that $s(t) = \int v(t)\,dt = \int 3t^2\,dt - \int 4t\,dt = \int 2\,dt = t^3 - 2t^2 + 2t + C$. By plugging in our initial value of $s(0) = 7$, we find that $7 = s(0) = C$ so that $s(t) = t^3 - 2t^2 + 2t + 7$. $\diamond$

Exercise 0.6.8. Consider tossing a ball upward with an initial velocity of 48 feet per second and constant acceleration of -32 feet per second from the edge of a cliff of height 432 feet. Compute the maximum height of the ball; then, find the time it takes for the ball to reach the ground.

0.7 Computing Area Bounded by a Curve of One Variable

Continuing in the theme of extrapolating data from intermittent observations, suppose that we observe the velocity $v(t)$ of a particle over a period of time $0 \leq t \leq 25$, taking care to mark down the velocity of the particle every five seconds. Consider along these lines the following table.

t	0	5	10	15	20	25
$v(t)$	25	31	35	43	47	46

We can roughly approximate the total distance traveled by the body for $0 \leq t \leq 25$ by assuming (incorrectly) that the body maintains a constant velocity each time we see it. Computing the total distance travelled by the particle during our observation amounts to finding the **displacement** of the body over each time interval and adding these quantities together. Explicitly, we have that

$$\text{total distance traveled} = 25 \cdot 5 + 31 \cdot 5 + 35 \cdot 5 + 43 \cdot 5 + 47 \cdot 5 + 46 \cdot 5 = 1135.$$

Certainly, we can improve this estimation by taking more measurements: even recording one more observation will give us a better understanding of the behavior of the particle over the specified interval of time. Better yet, the more observations we record, the more accurate our understanding of the total distance traveled; however, this also requires adding more numbers together. Consequently, it will be convenient to develop notation to take sums of arbitrarily large quantities of data.

Let us assume for the moment that we have a collection of n real numbers $a_1, a_2, \dots, a_n$ for some positive integer n. Certainly, the sum of these real numbers can be realized as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

We refer to this as **sigma notation**: indeed, the Greek letter sigma Σ is used as a mnemonic device for "sum"; the subscript $i = 1$ denotes the **index of summation** and informs us of the first term a_1 in our collection of data; and the superscript n tells us that the sum terminates with the last term a_n in our collection of data. We refer to the real number a_i as the ith **summand** for each integer $1 \leq i \leq n$; the entire sum $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called a **finite sum**.

Often, we will consider finite sums whose ith summand can be conveniently expressed in **closed-form**. Explicitly, this means that there exists a function $f(x)$ such that $a_i = f(i)$.

Example 0.7.1. Consider the finite sum $1 + 2 + 3 + \cdots + 10$ of the first ten positive integers. Observe that the ith summand is simply the positive integer i, hence we have that $a_i = i$ and

$$1 + 2 + 3 + \cdots + 10 = \sum_{i=1}^{10} i.$$

Crucially, we point out another way to **index** the given sum — namely, we have that

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + \cdots + 10 = 0 + 1 + 2 + 3 + \cdots + 10 = \sum_{i=0}^{10} i.$$

Often, if a sum involves a summand of zero, we will simply omit it (unless it is more convenient to include it). We could have also written this sum in a third way as follows.

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + \cdots + 10 = (1 + 2 + 3 + \cdots + 20) - (11 + 12 + 13 + \cdots + 20) = \sum_{i=1}^{20} i - \sum_{i=11}^{20} i.$$

Example 0.7.2. Consider the finite sum $1 + 4 + 9 + \cdots + 100$ of squares of the first ten positive integers in which the ith summand is simply the positive integer i^2. We have that $a_i = i^2$ and

$$1 + 4 + 9 + \cdots + 100 = \sum_{i=1}^{10} i^2.$$

Example 0.7.3. Express the finite sum $1^3 + 2^3 + 3^3 + \cdots + 1000^3$ of cubes of the first 1000 positive integers in summation notation, identifying the closed-form expression for the ith summand a_i.

Quite importantly, finite sums admit a convenient arithmetic of their own.

Proposition 0.7.4 (Properties of Finite Sums)**.** *Given any positive integer n and any real numbers $a_1, \dots, a_n, b_1, \dots, b_n$, and C, the following identities hold.*

(i.) (Empty Sum Law) *We have that $\sum_{i=n}^m a_i = 0$ for all integers $m < n$.*

(ii.) (Constant Sum Formula) *We have that $\sum_{i=m}^n C = C(n - m + 1)$ for all integers $m \leq n$.*

(iii.) (Linearity of a Finite Sum I) *We have that $\sum_{i=1}^n Ca_i = C(\sum_{i=1}^n a_i)$.*

(iv.) (Linearity of a Finite Sum II) *We have that $\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$.*

One can easily prove the above formulas by expanding and comparing the expressions on both sides of the equation. We will not endeavor to prove the following identities because these details are beyond the scope of this course; however, they will be indispensable in what follows.

Proposition 0.7.5. *Consider any positive integer n.*

(i.) *We have that* $\displaystyle\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

(ii.) *We have that* $\displaystyle\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

(iii.) *We have that* $\displaystyle\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Going back to our example of tracking a particle over a period of time, if we know the velocity $v(t)$ of the particle at any time $0 \leq t \leq 25$, then we can approximate the total distance traveled by the particle by recording the velocity a positive integer n times and computing the total displacement of the particle over each interval of time. Explicitly, if we observe the particle for some real numbers $0 = t_0 < t_1 < \cdots < t_n = 25$ and we assume that the particle has constant velocity $v(t_i)$ for each integer $0 \leq i \leq n$, then the total distance traveled by the particle between time t_{i-1} and time t_i is given by the real number $\Delta t_i = t_i - t_{i-1}$ and the total displacement of the particle on this closed interval $[t_{i-1}, t_i]$ is $v(t_i)\,\Delta t_i$ (rate $\times$ time). Consequently, in sigma notation, we have that

$$\text{total distance traveled} = v(t_1)\,\Delta t_1 + v(t_2)\,\Delta t_2 + \cdots + v(t_n)\,\Delta t_n = \sum_{i=1}^n v(t_i)\,\Delta t_i.$$

By viewing the points $(t_i, v(t_i))$ as lying on the graph of the velocity curve $v(t)$, we may recognize $\sum_{i=1}^n v(t_i)\,\Delta t_i$ as an approximation of the area between the curve $v(t)$ and the t-axis, i.e., the net area bounded by the curve $v(t)$ of one variable. We will now generalize this idea.

Consider any real function $f(x)$ that is continuous on a closed and bounded interval $[a, b]$. Choose any positive integer n; then, choose n real numbers $a = x_0 < x_1 < \cdots < x_n = b$. Consider the closed and bounded intervals $[x_{i-1}, x_i]$ for each integer $1 \leq i \leq n$. We refer to the collection $\mathcal{P}$ of such closed and bounded intervals as a **partition** of $[a, b]$, and we denote by $\Delta x_i = x_i - x_{i-1}$ the length of the interval $[x_{i-1}, x_i]$. Choosing **sample points** x_i^* such that $x_{i-1} \leq x_i^* \leq x_i$ yields a so-called **tagged partition** $(\mathcal{P}, x_i^*)$ consisting of closed and bounded intervals and sample points within them. We associate to each tagged partition a **Riemann sum** (or **Riemann approximation**)

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n.$$

Geometrically, we may realize $f(x_i^*)$ as the height of a rectangle with base Δx_i, hence the above Riemann sum provides an approximation of the **net area** bounded by the curve $f(x)$ over the closed interval $[a, b]$. Common tagged partitions are formed by taking x_i^* to be the left- or right-**endpoint** or the **midpoint** of $[x_{i-1}, x_i]$. Each of these tagged partitions uses $n + 1$ **equally-spaced** points $a = x_0 < x_1 < \cdots < x_n = b$; the common length of each interval $[x_{i-1}, x_i]$ is Δx. Considering that

$$b - a = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = \sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} \Delta x = n\,\Delta x$$

by the second part of Proposition 0.7.4, we conclude that $\Delta x = \frac{b-a}{n}$.

- We denote by $\mathcal{L}_n$ the **left-endpoint Riemann approximation** of the function $f(x)$ on the closed interval $[a, b]$ with $\Delta x_i = \Delta x = \frac{b-a}{n}$ and sample points $x_i^* = \ell_i = a + (i-1)\,\Delta x$.
- We denote by $\mathcal{R}_n$ the **right-endpoint Riemann approximation** of the function $f(x)$ on the closed interval $[a, b]$ with $\Delta x_i = \Delta x = \frac{b-a}{n}$ and sample points $x_i^* = r_i = a + i\,\Delta x$.
- We denote by $\mathcal{M}_n$ the **midpoint Riemann approximation** of the function $f(x)$ on the closed interval $[a, b]$ with $\Delta x_i = \Delta x = \frac{b-a}{n}$ and sample points $x_i^* = m_i = a + \frac{2i-1}{2}\,\Delta x$.

Example 0.7.6. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $f(x) = x$ on the closed and bounded interval $[0, 4]$ using four equally-spaced points.

Solution. By recognizing that $a = 0$ and $b = 4$, the length of each interval of the partition is

$$\Delta x_i = \Delta x = \frac{4-0}{4} = \frac{4}{4} = 1.$$

Consequently, the left-endpoint approximation satisfies that $\ell_i = 0 + (i-1)1 = i - 1$; the right-endpoint approximation satisfies that $r_i = 0 + i = i$; and the midpoint approximation satisfies that $m_i = 0 + \frac{2i-1}{2}(1) = \frac{2i-1}{2}$ for each integer $1 \leq i \leq 4$. We conclude therefore that the following hold.

$$\mathcal{L}_4 = \sum_{i=1}^{4} f(\ell_i)\,\Delta x = \sum_{i=1}^{4} \ell_i = \sum_{i=1}^{4} (i-1) = \sum_{i=1}^{4} i - \sum_{i=1}^{4} 1 = \frac{4(4+1)}{2} - 4 = 6$$

$$\mathcal{R}_4 = \sum_{i=1}^{4} f(r_i)\,\Delta x = \sum_{i=1}^{4} r_i = \sum_{i=1}^{4} i = \frac{4(4+1)}{2} = 10$$

$$\mathcal{M}_4 = \sum_{i=1}^{4} f(m_i)\,\Delta x = \sum_{i=1}^{4} \frac{2i-1}{2} = \sum_{i=1}^{4} i - \sum_{i=1}^{4} \frac{1}{2} = 10 - \frac{1}{2}(4) = 8$$

◇

Example 0.7.7. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $g(x) = x^2$ on the closed and bounded interval $[0, 1]$ using five equally-spaced points.

Solution. Like before, we find that $a = 0$ and $b = 1$ so that the length of each interval is

$$\Delta x = \frac{1-0}{5} = \frac{1}{5}.$$

By the above, the left-endpoint approximation uses the sample points $\ell_i = 0 + (i-1)\,\Delta x = \frac{i-1}{5}$; the right-endpoint approximation uses the sample points $r_i = 0 + i\Delta x = \frac{i}{5}$; and the midpoint approximation uses the sample points $m_i = 0 + \frac{2i-1}{2}\Delta x = \frac{2i-1}{10}$. We conclude that

$$\mathcal{L}_5 = \sum_{i=1}^{5} g(\ell_i)\,\Delta x = \sum_{i=1}^{5} \frac{\ell_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{i-1}{5}\right)^2 = \frac{1}{5}\left(0 + \frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25}\right) = \frac{30}{75} = \frac{6}{25}$$

$$\mathcal{R}_5 = \sum_{i=1}^{5} g(\ell_i)\,\Delta x = \sum_{i=1}^{5} \frac{r_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{i}{5}\right)^2 = \frac{1}{5}\left(\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + \frac{25}{25}\right) = \frac{55}{75} = \frac{11}{25}$$

$$\mathcal{M}_5 = \sum_{i=1}^{5} g(m_i)\,\Delta x = \sum_{i=1}^{5} \frac{m_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{2i-1}{10}\right)^2 = \frac{1}{5}\left(\frac{1}{100} + \frac{9}{100} + \frac{25}{100} + \frac{49}{100} + \frac{81}{100}\right) = \frac{33}{100} \quad \diamond$$

Exercise 0.7.8. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $h(x) = x^3$ on the closed and bounded interval $[0, 2]$ using eight equally-spaced points.

By allowing the number of sample points to grow arbitrarily large, the error of approximating the area bounded by a curve of one variable by a Riemann sum shrinks to zero, hence we define

$$\text{area bounded by the curve } f(x) \text{ on the closed and bounded interval } [a, b] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i,$$

where x_i^* are sample points of a partition $\mathcal{P}$ of $[a, b]$ and $\Delta x_i = x_i - x_{i-1}$ for each integer $1 \leq i \leq n$.

Example 0.7.9. Compute the area bounded by $f(x) = x^2$ on the closed interval $[0, 1]$.

Solution. Crucially, the above definition of the area does not depend on the choice sample points x_i^* or the partition $\mathcal{P}$ of $[0, 1]$, so we may carefully construct these to make things as convenient as possible. Given any choice of equally-spaced points $a = x_0 < x_1 < \cdots < x_n = b$, we have that $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. We may choose the right-endpoint approximation so that $x_i^* = \frac{i}{n}$ and

$$\mathcal{R}_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\frac{i^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

by the second part of Proposition 0.7.5. By taking the limit as $n \to \infty$, we conclude that

$$\text{area bounded by } x^2 \text{ on } [0, 1] = \lim_{n\to\infty}\mathcal{R}_n = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \frac{1}{3}. \quad \diamond$$

0.8 Definite Integration

Given any real function $f(x)$ and any real numbers a and b, consider any collection of points $(x_n, f(x_n))$ on the graph of $f(x)$ with $a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ for each integer $1 \leq i \leq n$. Each of the closed and bounded intervals $[x_{i-1}, x_i]$ gives rise to a partition $\mathcal{P}$ of the closed interval $[a, b]$, and we may choose sample points x_i^* for each integer $1 \leq i \leq n$ such that $x_{i-1} \leq x_i^* \leq x_i$ and $x_1^* < x_2^* < \cdots < x_n^*$. Crucially, we are not assuming here that the points $x_0, x_1, \ldots, x_n$ are equally-spaced, hence we may denote $\|\mathcal{P}\| = \max\{\Delta x_i \mid 1 \leq i \leq n\}$. We define

$$\int_a^b f(x)\,dx = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n f(x_i^*)\,\Delta x_i$$

as the **definite integral** of $f(x)$ on the closed and bounded interval $[a, b]$. Provided that the above limit exists, we say that $f(x)$ is **integrable** on $[a, b]$. We refer to the function $f(x)$ in this case as the **integrand**; the real numbers a and b are the **limits of integration**. By our work in the previous section, we may interpret the definite integral $\int_a^b f(x)\,dx$ as the net area bounded by $f(x)$: indeed, $\sum_{i=1}^n f(x_i^*)\,\Delta x_i$ is a Riemann sum representing rectangles of height $f(x_i^*)$ and width Δx_i.

Example 0.8.1. Express the following as the definite integral of a function on the interval $[1, 8]$.

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n \sqrt{2x_i^* + (x_i^*)^2}\,\Delta x_i$$

Solution. Considering that we do not know the partition $\mathcal{P}$ or the sample points x_i^*, there is not much we can do other than recognize the function $f(x)$. Comparing the limit with the definition above, we recognize that $f(x) = \sqrt{2x + x^2}$ so that the limit in question is $\int_1^8 \sqrt{2x + x^2}\,dx$. ⋄

Exercise 0.8.2. Express the following as the definite integral of a function on the interval $[0, \pi]$.

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n x_i^* \sin(x_i^*)\,\Delta x_i$$

Often, it is most simple to work with a **regular partition** $\mathcal{P}$, i.e., a partition of $[a, b]$ with $n+1$ equally-spaced points $a = x_0 < x_1 < \cdots < x_n = b$ such that $\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \Delta x = \frac{b-a}{n}$. Under this identification, we have that $\Delta x_1 = x_1 - x_0$ so that $x_1 = x_0 + \Delta x_1 = a + \Delta x$, from which it follows that $x_2 = x_1 + \Delta x_2 = (a + \Delta x) + \Delta x = a + 2\Delta x$ and $x_i = a + i\Delta x$ for each integer $1 \leq i \leq n$. Choosing our sample points such that $x_i^* = x_i = a + i\Delta x$ and using the fact that

$$\|\mathcal{P}\| = \max\{\Delta x_i \mid 1 \leq i \leq n\} = \Delta x = \frac{b-a}{n}$$

approaches zero if and only if n approaches ∞, we conclude that

$$\int_a^b f(x)\,dx = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n f(x_i^*)\,\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^n f(a + i\Delta x)\left(\frac{b-a}{n}\right).$$

Example 0.8.3. Express the following as the definite integral of a function on a closed interval.

$$\lim_{n \to \infty} \sum_{i=1}^n \cos\left(-\pi + i\frac{2\pi}{n}\right)\left(\frac{2\pi}{n}\right)$$

Solution. Considering that $\Delta x = \frac{2\pi}{n} = \frac{b-a}{n}$ and $a = -\pi$, we must have $b = a + n\Delta x = -\pi + 2\pi = \pi$. Even more, the integrand is $\cos(x)$, hence the limit describes the quantity $\int_{-\pi}^{\pi} \cos(x)\, dx$. ⋄

Before we endeavor to compute any definite integrals by the limit definition provided above, it is conceptually important to note that the definite integral can be computed by hand in some cases without appealing to any limits. Explicitly, for any real numbers c and d, we have that $\int_a^b (cx+d)\, dx$ represents the net area bounded by the line $cx + d$ and the coordinate axes. Consequently, this area can be computed geometrically as a linear combination of areas of triangles and rectangles.

Exercise 0.8.4. Compute the definite integral $\int_{-2}^{3} (3x-2)\, dx$ using geometry.

Exercise 0.8.5. Compute the definite integral $\int_{-3}^{2} (5-2x)\, dx$ using geometry.

Likewise, for any function of the form $y = f(x) = \sqrt{r^2 - x^2}$, it follows that $x^2 + y^2 = r^2$ yields a circle of radius r, hence we can determine an integral of the form $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$.

Exercise 0.8.6. Compute the definite integral $\int_{-1}^{1} \sqrt{1-x^2}\, dx$ using geometry.

Often, we will deal with definite integrals that cannot be computed by geometry; for now, if we encounter this situation, we can sometimes use the limit definition of the definite integral.

Example 0.8.7. Compute the definite integral $\int_0^1 x^2\, dx$ as the limit of a Riemann approximation as the number n of subintervals tends to infinity.

Solution. Considering that $a = 0$ and $b = 1$, we have that

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

so that $a + i\Delta x = 0 + \frac{i}{n} = \frac{i}{n}$. Consequently, it follows that

$$\int_0^1 x^2\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n \to \infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}.$$ ⋄

Example 0.8.8. Compute the definite integral $\int_0^3 (x^3 - 6x)\, dx$ as the limit of a Riemann approximation as the number n of subintervals tends to infinity.

Solution. Considering that $a = 0$ and $b = 3$, we have that

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

so that $a + i\Delta x = 0 + \frac{3i}{n} = \frac{3i}{n}$. Consequently, it follows that

$$\int_0^3 (x^3 - 6x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\left(\frac{3}{n}\right) = \lim_{n \to \infty} \frac{3}{n^2} \sum_{i=1}^{n} \left(\frac{27i^3}{n^2} - 18i\right).$$

Granted that the limit of each of these Riemann sums exists, the limit of their difference is given by the difference of their limits, hence it suffices to compute these limits separately.

$$\lim_{n \to \infty} \frac{3}{n^2} \sum_{i=1}^{n} \frac{81i^3}{n^2} = \lim_{n \to \infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n \to \infty} \frac{81}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 = \frac{81}{4}$$

$$\lim_{n \to \infty} \frac{3}{n^2} \sum_{i=1}^{n} 18i = \lim_{n \to \infty} \frac{54}{n^2} \sum_{i=1}^{n} i = \lim_{n \to \infty} \frac{54}{n^2} \cdot \frac{n(n+1)}{2} = \frac{54}{2} = \frac{108}{4}$$

Consequently, we have that $\int_0^3 (x^3 - 6x)\,dx = \frac{81}{4} - \frac{108}{4} = -\frac{27}{4}$. ◇

Based on the definition of the definite integrals and the summation properties outlined in the previous section, we can extrapolate the following properties of definite integrals.

Proposition 0.8.9 (Properties of Definite Integrals)**.** *Given any real function $f(x)$ that is integrable on a closed and bounded interval $[a, b]$, the following properties hold for $\int_a^b f(x)\,dx$.*

(i.) (Empty Integral Law) $\int_a^a f(x)\,dx = 0$

(ii.) (Reversing the Limits of Integration) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(iii.) (Additivity of Adjacent Intervals) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *for all real numbers* c

(iv.) (Constant Integral Formula) $\int_a^b k\,dx = k(b-a)$ *for all real numbers* k

(v.) (Linearity of a Definite Integral I) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ *for all real numbers* k

(vi.) (Linearity of a Definite Integral II) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Example 0.8.10. Compute the definite integral $\int_0^1 (3x^2 + 4)\,dx$.

Solution. By appealing to Example 0.8.7 and Proposition 0.8.9, we have that

$$\int_0^1 (3x^2 + 4)\,dx = \int_0^1 3x^2\,dx + \int_0^1 4\,dx = 3\int_0^1 x^2\,dx + 4(1-0) = 3\left(\frac{1}{3}\right) + 4 = 5.$$ ◇

Example 0.8.11. Given any pair of real functions $f(x)$ and $g(x)$ such that $\int_{-1}^1 f(x)\,dx = 2$ and $\int_{-1}^1 g(x)\,dx = -1$, compute the definite integral $\int_{-1}^1 [3f(x) - g(x)]\,dx$.

Solution. By appealing to Proposition 0.8.9, we have that

$$\begin{aligned}\int_{-1}^1 [3f(x) - g(x)]\,dx &= \int_{-1}^1 (3f(x) + [-g(x)])\,dx \\ &= \int_{-1}^1 3f(x)\,dx + \int_{-1}^1 [-g(x)]\,dx \\ &= 3\int_{-1}^1 f(x)\,dx - \int_{-1}^1 g(x)\,dx = 3(2) - (-1) = 7.\end{aligned}$$ ◇

Example 0.8.12. Given any real function $f(x)$ such that $\int_0^4 f(x)\,dx = 1$, $\int_{-2}^3 f(x)\,dx = 3$, and $\int_{-2}^0 f(x)\,dx = 5$, compute the definite integral $\int_3^4 f(x)\,dx$.

Solution. By appealing to Proposition 0.8.9, we have that

$$\int_3^4 f(x)\,dx = \int_3^{-2} f(x)\,dx + \int_{-2}^4 f(x)\,dx$$

$$= -\int_{-2}^3 f(x)\,dx + \int_{-2}^4 f(x)\,dx$$

$$= -\int_{-2}^3 f(x)\,dx + \int_{-2}^0 f(x)\,dx + \int_0^4 f(x)\,dx = -3 + 5 + 1 = 3. \qquad \diamond$$

0.9 The Fundamental Theorem of Calculus

Calculus can be divided into two topics — differentiation and integration — that are connected by the Fundamental Theorem of Calculus. Essentially, the Fundamental Theorem of Calculus says that differentiation and integration are inverse operations: if $f(x)$ is continuous on an open interval, then $f(x)$ admits an antiderivative by the definite integral, and conversely, the definite integral of $f(x)$ over a closed interval measures the **net change** of any antiderivative over that interval.

Theorem 0.9.1 (Fundamental Theorem of Calculus, Part I)**.** *Given any real function $f(x)$ that is integrable with a continuous antiderivative $F(x)$ on a closed interval $[a, b]$, we have that*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Even more, this quantity measures the net area bounded by the curve $f(x)$ from $x = a$ to $x = b$.

Proof. Observe that the quantity $F(b)-F(a)$ measures the net change of $F(x)$ on the closed interval $[a, b]$. Given any collection of n real numbers $a = x_0 < x_1 < \cdots < x_n = b$, we have that

$$F(b) - F(a) = F(b) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_1) - F(a)$$

by adding and subtracting $F(x_i)$ for each integer $1 \leq i \leq n-1$. Grouping each consecutive pair of differences and using the fact that $a = x_0$ and $b = x_n$, it follows that

$$F(b) - F(a) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})].$$

By the Mean Value Theorem applied to $F(x)$, for each integer $1 \leq i \leq n$, there exists a real number x_i^* such that $x_{i-1} \leq x_i^* \leq x_i$ and $F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1})$. By assumption that $F(x)$ is an antiderivative of $f(x)$ on the closed interval $[a, b]$, we have that $F'(x) = f(x)$, hence we can rewrite each of these equations as $F(x_i) - F(x_{i-1}) = f(x_i^*)\,\Delta x_i$ for the quantity $\Delta x_i = x_i - x_{i-1}$. Going back to our above displayed equation with this new identity, we have that

$$F(b) - F(a) = \sum_{i=1}^n f(x_i^*)\,\Delta x_i.$$

By taking the limit as n approaches ∞ on both sides, we conclude the desired result that

$$F(b) - F(a) = \lim_{n \to \infty} [F(b) - F(a)] = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x_i = \int_a^b f(x) \, dx. \qquad \square$$

Consequently, if $v(t)$ measures the velocity of a particle over time, then the (definite) integral of $v(t)$ over $[a, b]$ measures the total distance travelled by the particle from time $t = a$ to time $t = b$.

Exercise 0.9.2. Compute the net area bounded by the curve $f(x) = x^3$ from $x = -1$ to $x = 1$.

Exercise 0.9.3. Compute the net area bounded by the curve $g(x) = \sin(x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

Exercise 0.9.4. Compute the net area bounded by the curve $h(x) = \frac{1}{x}$ from $x = 1$ to $x = e$.

Remark 0.9.5. Like we previously mentioned, if $F(x)$ is an antiderivative of a real function $f(x)$ on a closed interval $[a, b]$, the Mean Value Theorem implies that every antiderivative of $f(x)$ over $[a, b]$ is of the form $F(x) + C$ for some real number C. Consequently, the choice of antiderivative of $f(x)$ does not matter when it comes to computing the definite integral of $f(x)$ on $[a, b]$:

$$\int_a^b f(x) \, dx = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

holds for all real numbers C by the Fundamental Theorem of Calculus, Part I.

One other way to interpret the first part of the Fundamental Theorem of Calculus is as follows.

Corollary 0.9.6 (Net Change Theorem)**.** *Given any differentiable function $f(x)$ on an open interval (a, b) such that $f(a)$ and $f(b)$ are defined, we have that*

$$f(b) - f(a) = \int_a^b f'(x) \, dx.$$

Put another way, the net change of $f(x)$ over the closed interval $[a, b]$ is $\int_a^b f'(x) \, dx$.

Exercise 0.9.7. Consider a leaky water heater that loses $2 + 5t$ gallons of water per hour for each hour after 7 AM. Compute the total amount of water leaked between the time of 9 AM and 12 PM.

Exercise 0.9.8. Consider any medication that disperses into a patient's bloodstream at a rate of $50 - 2\sqrt{t}$ milligrams per hour from the time it is administered. Compute the amount of medication dispersed into a patient's bloodstream one hour after it is administered. Given that one full dose is 50 milligrams, what percentage of the dose reaches the patient's bloodstream in an hour?

Exercise 0.9.9. Consider any particle that moves with velocity $t^3 - 10t^2 + 24t$ meters per second after initial observation at time $t = 0$. Compute the total displacement of and the total distance travelled by the particle from time $t = 0$ to time $t = 6$; then, compare the values.

Conversely, the second part of the Fundamental Theorem of Calculus states that every continuous function on a closed interval $[a, b]$ admits an antiderivative in the form of a definite integral.

Theorem 0.9.10 (Fundamental Theorem of Calculus, Part II)**.** *Given any real function $f(x)$ that is continuous on a closed interval $[a, b]$, for all real numbers $a < x < b$, we have that*

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

Proof. Considering that $f(x)$ is continuous on $[a, b]$, it is integrable on $[a, b]$, hence we may define

$$F(x) = \int_a^x f(t)\,dt$$

for all real numbers $a \leq x \leq b$. We must demonstrate that for all real numbers $a < x < b$, the limit

$$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

exists. By the second and third parts of Proposition 0.8.9, it follows that

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

By the **Mean Value Theorem for Definite Integrals**, there exists a real number c (depending upon h) such that $x < c < x + h$ and $\int_x^{x+h} f(t)\,dt = f(c)[(x+h) - x] = f(c)h$ so that

$$f(c) = \frac{F(x+h) - F(x)}{h}.$$

Considering that $f(x)$ is continuous on the closed interval $[a, b]$, it follows that

$$f\left(\lim_{h \to 0} c\right) = \lim_{h \to 0} f(c) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = F'(x),$$

hence it suffices to compute the limit of c as h approaches 0. By the Squeeze Theorem, we have

$$x = \lim_{h \to 0} x \leq \lim_{h \to 0} c \leq \lim_{h \to 0} (x+h) = x$$

so that $\lim_{h \to 0} c = x$ and $F'(x) = f(x)$ for all real numbers $a < x < b$, as desired. □

Exercise 0.9.11. Compute the derivative of $\int_0^x \sin(t)\,dt$ for any real number $x > 0$.

Exercise 0.9.12. Compute the derivative of $\int_{-1}^x e^t\,dx$ for any real number $x > -1$.

Exercise 0.9.13. Compute the derivative of $\int_1^x \ln(t)\,dt$ for any real number $x > 1$.

Exercise 0.9.14. Given any differentiable real functions $f(x)$, $g(x)$, and $h(x)$, use the Fundamental Theorem of Calculus, Part II and the Chain Rule for derivatives to prove that

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Exercise 0.9.15. Compute the derivative of $\int_0^{x^2} \sin(\cos(t))\,dt$ for any real number $x > 0$.

Exercise 0.9.16. Compute the derivative of $\int_{\ln(x)}^{10} \sqrt{t^2+1}\,dt$ for any real number $0 < x < e^{10}$.

Exercise 0.9.17. Compute the derivative of $\int_{x^3}^{x^2} \sqrt{t}\,dt$ for any real number $0 < x < 1$.

0.10 u-Substitution

Until now, we have managed to find the antiderivatives of many functions by viewing antidifferentiation as the inverse to differentiation (in the sense of the Fundamental Theorem of Calculus, Part I) and subsequently using the appropriate analog of the familiar rules for differentiation such as the Power Rule and the Chain Rule. Explicitly, given any real number $r \neq -1$, we have that

$$\int x^r \, dx = \frac{1}{r+1} x^{r+1} + C$$

by the Power Rule. Further, for any differentiable functions $f(x)$ and $g(x)$, we have that

$$\int f'(g(x))g'(x) \, dx = f(g(x)) + C$$

by the Chain Rule. Essentially, if we make the assignment $u = g(x)$, then it follows that $\frac{du}{dx} = g'(x)$ and $f'(g(x))g'(x) = f'(u)\frac{du}{dx}$. Conventionally, this relationship is written as $du = g'(x) \, dx$ so that $f'(g(x))g'(x) \, dx = f'(u) \, du$. Considering that $f(u)$ is an antiderivative of $f'(u)$, it follows that

$$\int f'(g(x))g'(x) \, dx = \int f'(u) \, du = f(u) + C = f(g(x)) + C.$$

Colloquially, we refer to this technique (and its broader applications) as u-**substitution**.

Exercise 0.10.1. Compute the indefinite integral of $f(x) = (x+1)^{100}$.

Exercise 0.10.2. Compute the indefinite integral of $g(x) = x\cos(2x^2)$.

Exercise 0.10.3. Compute the indefinite integral of $h(x) = x^2 e^{x^3}$.

Exercise 0.10.4. Compute the indefinite integral of $k(x) = x\sqrt{2x-1}$.

Even more, the technique of u-substitution can be used to evaluate definite integrals. Explicitly, suppose that $f'(x)$ is integrable on the closed interval $[g(a), g(b)]$ and $f'(g(x))g'(x)$ is integrable on the closed interval $[a, b]$. By performing the substitution $u = g(x)$, we have that $du = g'(x) \, dx$ and $f'(g(x))g'(x) \, dx = f'(u) \, du$. Even more, if $x = a$, then $u = g(a)$, and if $x = b$, then $u = g(b)$ so that

$$\int_a^b f'(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f'(u) \, du.$$

Exercise 0.10.5. Compute the definite integral $\int_0^1 x^4(x^5-1)^{10} \, dx$.

Exercise 0.10.6. Compute the definite integral $\int_{-\pi/4}^{\pi/4} 2x\sec^2(x^2) \, dx$

Exercise 0.10.7. Compute the definite integral $\int_1^e \frac{\ln(x)}{x} \, dx$.

We say that a real function $f(x)$ is **even** if it holds that $f(-x) = f(x)$ for all real numbers x in the domain of f such that $-x$ is in the domain of f. Consequently, the polynomial $3x^4 - x^2 + 2$ and the trigonometric function $\cos(x)$ are even functions. Conversely, we say that $f(x)$ is **odd** if it holds that $f(-x) = -f(x)$ for all real numbers x in the domain of f such that $-x$ is in the domain of f. We note that the polynomial $4x^5 + x + 1$ and the trigonometric function $\sin(x)$ are odd functions. We refer to the property that a function is even or odd as the **parity** of the function. We note that a function need not have parity, as illustrated by the fact that $f(x) = x^2 + x$ does not satisfy either $f(-x) = f(x)$ or $f(-x) = -f(x)$; however, the parity of a function is always well-defined.

Exercise 0.10.8. Explain whether $f(x) = \tan(x)$ is even, odd, or neither.

Exercise 0.10.9. Explain whether $g(x) = x^2 e^x$ is even, odd, or neither.

Exercise 0.10.10. Explain whether $h(x) = \sin^2(x)$ is even, odd, or neither.

Proposition 0.10.11 (Properties of Function Parity)**.** *Consider any real functions $f(x)$ and $g(x)$.*

(i.) (Preservation of Parity Under Nonzero Scalar Multiple) *If $f(x)$ has parity, then for all nonzero real numbers α, the scalar multiple $\alpha f(x)$ of $f(x)$ by α has the same parity as $f(x)$.*

(ii.) (Preservation of Parity Under Sum) *If $f(x)$ and $g(x)$ have the same parity, then their sum $f(x) + g(x)$ has the same parity as both $f(x)$ and $g(x)$.*

(iii.) (Preservation of Parity Under Product) *If $f(x)$ and $g(x)$ have the same parity, then their product $f(x)g(x)$ has the same parity as both $f(x)$ and $g(x)$.*

(iv.) (Products of Functions of Opposite Parity) *If $f(x)$ and $g(x)$ have opposite parity, then their product $f(x)g(x)$ is an odd function.*

(v.) (Preservation of Parity Under Quotient) *If $f(x)$ and $g(x)$ have the same parity, then their quotient $f(x)/g(x)$ has the same parity as both $f(x)$ and $g(x)$.*

(vi.) (Quotients of Functions of Opposite Parity) *If $f(x)$ and $g(x)$ have opposite parity, then their quotient $f(x)/g(x)$ is an odd function.*

(vii.) (Preservation of Parity Under Composition) *If $f(x)$ and $g(x)$ have the same parity, then their composite $f(g(x))$ has the same parity as both $f(x)$ and $g(x)$.*

(viii.) (Composition of Functions of Opposite Parity) *If $f(x)$ and $g(x)$ have opposite parity, then their composite $f(g(x))$ is an even function.*

(ix.) (Parity of the Derivative of a Function) *If $f(x)$ is differentiable and $f(x)$ has parity, then the derivative $f'(x)$ has the opposite parity of $f(x)$.*

Proposition 0.10.12 (Definite Integral of an Even Function on a Symmetric Interval)**.** *Consider any even real function $f(x)$ that is integrable on a closed interval $[-a, a]$. We have that*

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Proof. By the third property of Proposition 0.8.9, it follows that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Consider the substitution $u = -x$ with $du = -dx$. By assumption that $f(-x) = f(x)$, we have that

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} f(-x)\,dx = \int_{a}^{0} f(u)(-du) = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

Consequently, by the above two displayed equations, the desired identity holds. □

Proposition 0.10.13 (Definite Integral of an Odd Function on a Symmetric Interval)**.** *Consider any odd real function $f(x)$ that is integrable on a closed interval $[-a, a]$. We have that*

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Proof. By the third property of Proposition 0.8.9, it follows that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

By assumption that $f(-x) = -f(x)$, the substitution $u = -x$ with $du = -dx$ yields that

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} -f(-x)\,dx = \int_{a}^{0} -f(u)(-du) = \int_{a}^{0} f(u)\,du = -\int_{0}^{a} f(u)\,du = -\int_{0}^{a} f(x)\,dx.$$

Consequently, by the above two displayed equations, the desired identity holds. □

0.11 Integration by Parts

We turn our attention next to an analog of the Product Rule for antidifferentiation. We adopt the shorthand notation $u = f(x)$ and $v = g(x)$ for some differentiable functions $f(x)$ and $g(x)$ so that $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$ or $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. By the Product Rule, we have that

$$\frac{d}{dx}[uv] = \frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Considering that uv is clearly an antiderivative of $\frac{d}{dx}[uv] = f(x)g'(x) + g(x)f'(x)$, it follows that

$$uv = \int [f(x)g'(x) + g(x)f'(x)]\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = \int u\,dv + \int v\,du.$$

By rearranging, we obtain an analog to the Product Rule for antidifferentiation.

Theorem 0.11.1 (Integration by Parts Formula)**.** *Given any differentiable functions $u = f(x)$ and $v = g(x)$, under the convention that $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we have that*

$$\int u\,dv = uv - \int v\,du.$$

Colloquially, we refer to this technique as the method of **integration by parts** because the rule allows us to identify two parts of the integrand — namely, u and dv — in such a way that

(i.) the antiderivative of u is difficult to determine and its derivative du is simpler;

(ii.) the antiderivative v of dv is readily obtained; and

(iii.) the antiderivative of $v\,du$ is known or can be found by the method of integration by parts.

Exercise 0.11.2. Use integration by parts to compute the antiderivative of $x\cos(x)$.

Exercise 0.11.3. Use integration by parts to compute the antiderivative of $\ln(x)$.

Exercise 0.11.4. Use integration by parts to compute the antiderivative of xe^x.

Once again, the advantage of the method of integration by parts is that it allows us to trade an expression $u\,dv$ that is difficult to antidifferentiate for an expression $v\,du$ whose antiderivative is known or can be found by integration by parts. Consequently, we may identify families of functions whose antiderivatives are unknown to us at this time — e.g., logarithmic and inverse trigonometric functions — and use these as candidates for u. On the other hand, we may identify functions whose antiderivatives are easily found — e.g., algebraic, trigonometric, and exponential functions — and use these as candidates for dv. Ultimately, this gives rise to the following acronym.

Logarithmic Inverse Trigonometric Algebraic Trigonometric Exponential

Essentially, this acronym is intended to help us remember how to prioritize the assignments of u and dv to our integrand: if the function is further left on the list, then it should be made u; if the function is further right on the list, it should be made dv. Consequently, we have the following.

Algorithm 0.11.5 (Using LIATE)**.** Given any pair of functions $f(x)$ and $g(x)$ such that

(a.) $f(x)$ is a logarithmic, inverse trigonometric, or algebraic function and

(b.) $g(x)$ is an algebraic, trigonometric, or exponential function,

in order to compute $\int f(x)g(x)\,dx$, we may assign $u = f(x)$ and $dv = g(x)\,dx$.

Exercise 0.11.6. Use integration by parts once to compute the antiderivative of $x^3 \ln(x)$.

Exercise 0.11.7. Use integration by parts twice to compute the antiderivative of $x^2 \sin(x)$.

Exercise 0.11.8. Use integration by parts three times to compute the antiderivative of $x^3 e^x$.

Exercise 0.11.9. Explain the difficulty in using integration by parts with $u = x^3$ and $dv = e^{x^2}\,dx$ to compute the antiderivative of $x^3 e^{x^2}$. Group the terms differently, and try again successfully.

Observe that in two of the above examples, we were required to use integration by parts multiple times in order to find the antiderivatives of the given functions. Generally, if we wish to evaluate the antiderivative of the product of a function $f(x)$ and a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$, we may use a shorthand version of integration by parts known as the **tabular method**.

Theorem 0.11.10 (Tabular Method for Integration)**.** *Given any function $f(x)$ whose antiderivatives are known and any polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ with nonzero a_n, we have that*

$$\int p(x)f(x)\,dx = \sum_{k=0}^{n} (-1)^k p^{(k)}(x) I^{k+1} f(x),$$

where $p^{(k)}(x)$ denotes the kth derivative of $p(x)$ and $I^k f(x)$ denotes the k-fold antiderivative of $f(x)$.

Proof. Observe that the nth derivative of $p(x)$ is given by $p^{(n)}(x) = a_n n!$ so that $p^{(n+1)}(x) = 0$. By the method of Integration by Parts Formula with $u = p(x)$ and $dv = f(x)\,dx$, we have that

$$\int p(x)f(x)\,dx = p(x)F(x) - \int p'(x)F(x)\,dx$$

for some real function $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$. By hypothesis, the antiderivative of $F(x)$ is known, hence we may use integration by parts with $u = p'(x)$ and $dv = F(x)\,dx$ to find that

$$\int p'(x)F(x)\,dx = p'(x)I^2 f(x) - \int p''(x)I^2 f(x)\,dx,$$

where $I^2 f(x)$ denotes the antiderivative of $F(x)$, i.e., $\frac{d}{dx}I^2 f(x) = F(x)$ so that $\frac{d^2}{dx^2}I^2 f(x) = f(x)$. Combined with the above displayed equation, we have that

$$\int p(x)f(x)\,dx = p(x)F(x) - \left(p'(x)I^2 f(x) - \int p''(x)I^2 f(x)\,dx\right)$$

$$= p(x)F(x) - p'(x)I^2 f(x) + \int p''(x)I^2 f(x)\,dx.$$

Using integration by parts once again with $u = p''(x)$ and $dv = I^2 f(x)\,dx$, we have that

$$\int p''(x)I^2 f(x)\,dx = p''(x)I^3 f(x) - \int p'''(x)I^3 f(x)\,dx.$$

Combined with the above displayed equation, we find that

$$\int p(x)f(x)\,dx = p(x)F(x) - p'(x)I^2 f(x) + p''(x)I^3 f(x) - \int p'''(x)I^3 f(x)\,dx.$$

Continue in this manner until $u = p^{(n)}(x)$. By our opening remarks, we have that $du = p^{(n+1)}(x) = 0$ so that $\int v\,du = 0$. Observing the pattern and using $F(x) = \int f(x)\,dx = I^1 f(x)$, we are done. □

Graphically, we can quite simply implement the tabular method by writing out a table with four columns: the first column consists of the index k; the second column consists of the sign $(-1)^k$; the third column consists of the consecutive derivatives of $p(x)$ up to and including 0; and the fourth column consists of the consecutive antiderivatives $I^{k+1} f(x)$ of $f(x)$. Once we have these, the tabular method guarantees that $\int p(x)f(x)\,dx$ can be found by adding the consecutive products of the kth row of the second and third columns by the $(k+1)$th row of the fourth column.

Example 0.11.11. We will illustrate the tabular method to compute the antiderivative of $x^2 \sin(x)$ as in Example 0.11.7. Construct the following table with $p(x) = x^2$ and $f(x) = \sin(x)$.

k	$(-1)^k$	$p^{(k)}(x)$	$I^{k+1} f(x)$
0	$+$	x^2	$\sin(x)$
1	$-$	$2x$	$-\cos(x)$
2	$+$	2	$-\sin(x)$
3	$-$	0	$\cos(x)$

Consequently, we find that $\int x^2 \sin(x)\,dx = x^2(-\cos(x)) - 2x(-\sin(x)) + 2\cos(x)$, as desired.

Exercise 0.11.12. Use the tabular method to verify your solution to Example 0.11.8.

Exercise 0.11.13. Use the tabular method to compute the antiderivative of $x^{10}(2x+1)^4$.

0.12 Trigonometric Integrals

Given positive integers (or whole numbers) m and n, we refer to an integral of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

as a **trigonometric integral**: indeed, the integrand is a product of powers of basic trigonometric functions. Quickly, one can glean that u-substitution fails, and integration by parts is hopelessly complicated. Using basic trigonometry, however, we are able to evaluate these integrals by converting them to a form in which we can use the tried-and-true methods of yore. Given a right triangle with hypotenuse of length $h > 0$, base of length a, and height of length o, the Pythagorean Theorem states that $o^2 + a^2 = h^2$. By dividing each term in this equation by h, we have that $\frac{o^2}{h^2} + \frac{a^2}{h^2} = 1$. Using x to represent the angle whose opposite side has length o and whose adjacent side has length a, the Pythagorean Theorem yields the so-called **Pythagorean Identity**

$$\sin^2(x) + \cos^2(x) = 1.$$

Consequently, we may convert any even power of $\cos(x)$ into a power of $1-\sin^2(x)$ (and vice-versa). Considering that $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$, we have the following stratagem.

Strategy 0.12.1 (Trigonometric Integration, Case I)**.** Consider the case that either m or n is odd.

(a.) Given that m is odd, we may write $m = 2k+1$ for some positive integer k so that

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^{2k+1}(x)\cos^n(x)\,dx = \int [\sin^2(x)]^k \cos^n(x)(\sin(x)\,dx).$$

Considering that $\sin^2(x) = 1-\cos^2(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$, letting $u = \cos(x)$ yields that

$$\int \sin^m(x)\cos^n(x)\,dx = -\int (1-u^2)^k u^n\,du.$$

Expanding the polynomial $(1-u^2)^k$ and using the Power Rule, we can find the antiderivative.

(b.) Given that n is odd, we may write $n = 2\ell+1$ for some positive integer ℓ so that

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^m(x)\cos^{2\ell+1}(x)\,dx = \int \sin^m(x)(\cos^2(x))^\ell(\cos(x)\,dx).$$

Considering that $\cos^2(x) = 1-\sin^2(x)$ and $\frac{d}{dx}\sin(x) = \cos(x)$, letting $v = \sin(x)$ yields that

$$\int \sin^m(x)\cos^n(x)\,dx = \int v^m(1-v^2)^\ell\,dv.$$

Expanding the polynomial $(1-v^2)^\ell$ and using the Power Rule, we can find the antiderivative.

Example 0.12.2. Compute the indefinite integral of $\sin^3(x)\cos^2(x)$.

Solution. Observe that $\sin^3(x)\cos^2(x)\,dx = \sin^2(x)\cos^2(x)(\sin(x)\,dx)$. By the Pythagorean Identity, we have that $\sin^2(x) = 1 - \cos^2(x)$, from which it follows that

$$\sin^3(x)\cos^2(x)\,dx = (1-\cos^2(x))\cos^2(x)(\sin(x)\,dx).$$

Using the substitution $u = \cos(x)$, we have that $du = -\sin(x)\,dx$ so that

$$\sin^3(x)\cos^2(x)\,dx = (1-u^2)u^2(-du) = (u^2-1)u^2\,du = (u^4-u^2)\,du.$$

Consequently, we find that

$$\int \sin^3(x)\cos^2(x)\,dx = \int (u^4-u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C. \qquad \diamond$$

Exercise 0.12.3. Compute the indefinite integral of $\sin^5(x)$.

Unfortunately, this method fails in the case that both m and n are even. Consider the trigonometric integral of $\sin^2(x)\cos^2(x)$. By setting $u = \sin(x)$, we find that $du = \cos(x)\,dx$ so that

$$\sin^2(x)\cos^2(x)\,dx = u^2\cos(x)\,du.$$

But the lingering factor of $\cos(x)$ obstructs our efforts to take the indefinite integral. Likewise, a similar obstruction appears if we attempt to let $u = \cos(x)$. Luckily, we have more trigonometric tools at our disposal. Recall the following **angle addition formulas**.

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Using these, we can derive the **double-angle formulas** by plugging in $x = y$.

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Considering that $\sin^2(x) + \cos^2(x) = 1$, we may simplify these identities as follows.

$$\cos(2x) = \cos^2(x) - \sin^2(x) = [1-\sin^2(x)] - \sin^2(x) = 1 - 2\sin^2(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1-\cos^2(x)) = 2\cos^2(x) - 1$$

By solving for $\sin^2(x)$ and $\cos^2(x)$ above, we obtain the **power-reduction formulas**.

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad\qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

One way to memorize the distinction is to "remember your sign" when using sine. Or as my former student Ronald Heminway so eloquently put it, we may use the mnemonic device "sinus minus."

Strategy 0.12.4 (Trigonometric Integration, Case II)**.** Consider the case that neither of the integers m and n is odd. Put another way, consider the case that both of the integers m and n are even.

(a.) Given that $m = n = 2k$ for some positive integer k, we have that

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^{2k}(x)\cos^{2k}(x)\,dx = \int [\sin(x)\cos(x)]^{2k}\,dx.$$

Using the double-angle formula $\sin(2x) = 2\sin(x)\cos(x)$, we have that

$$[\sin(x)\cos(x)]^{2k} = \left[\frac{\sin(2x)}{2}\right]^{2k} = \frac{[\sin^2(2x)]^k}{4^k}.$$

Using the power-reduction formula $\sin^2(2x) = \frac{1}{2}[1-\cos(4x)]$, we can then obtain a polynomial in $\cos(4x)$. Continue using the power-reduction formula for cosine to obtain a linear combination of $\cos(4x)$, $\cos(8x)$, $\cos(16x)$, etc. Each of these has an elementary antiderivative.

(b.) Given that $m = 2i$ and $n = 2j$ for some distinct positive integers i and j, use the power-reduction formulas repeatedly to express $\sin^{2i}(x)\cos^{2j}(x) = [\sin^2(x)]^i[\cos^2(x)]^j$ as a linear combination of $\cos(2x)$, $\cos(4x)$, $\cos(8x)$, etc. Each of these has an elementary antiderivative.

Example 0.12.5. Compute the indefinite integral of $\sin^2(x)\cos^2(x)$.

Solution. By the double-angle formula, we have that

$$\sin^2(x)\cos^2(x)\,dx = [\sin(x)\cos(x)]^2\,dx = \left(\frac{1}{2}\sin(2x)\right)^2 dx = \frac{1}{4}\sin^2(2x)\,dx.$$

Using the power-reduction formula, we find that

$$\sin^2(x)\cos^2(x)\,dx = \frac{1}{4}\sin^2(2x)\,dx = \frac{1}{4}\cdot\frac{1}{2}[1-\cos(4x)]\,dx$$

has an elementary antiderivative. Consequently, we conclude that

$$\int \sin^2(x)\cos^2(x)\,dx = \frac{1}{8}\int [1-\cos(4x)]\,dx = \frac{1}{8}\left[x - \frac{1}{4}\sin(4x)\right] + C. \qquad \diamond$$

Exercise 0.12.6. Compute the indefinite integral of $\cos^4(x)$.

Using the Pythagorean Identity $\sin^2(x) + \cos^2(x) = 1$, we can obtain another identity

$$\tan^2(x) + 1 = \sec^2(x)$$

by dividing each term by $\cos^2(x)$ and recalling that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\sec(x) = \frac{1}{\cos(x)}$. Consequently, we can adapt our stratagem from Trigonometric Integration, Case I to evaluate integrals of the form

$$\int \tan^m(x)\sec^n(x)\,dx.$$

Crucially, toward achieving this end, we must observe the following facts.

1.) By the Quotient Rule, we have that

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos^2(x) - (-\sin(x))(\sin(x))}{\cos^2(x)} = \frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)} = \sec^2(x).$$

2.) By the Chain Rule, we have that

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}[\cos(x)]^{-1} = -[\cos(x)]^{-2}[-\sin(x)] = \frac{\sin(x)}{\cos^2(x)} = \sec(x)\tan(x).$$

3.) Using the substitution $u = \cos(x)$ with $du = -\sin(x)\,dx$, we have that

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$$

4.) Using the substitution $u = \sec(x) + \tan(x)$ with $du = [\sec(x)\tan(x) + \sec^2(x)]\,dx$, we have

$$\int \frac{\sec(x)[\sec(x) + \tan(x)]}{\sec(x) + \tan(x)}\,dx = \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\tan(x) + \sec(x)}\,dx = \ln|\sec(x) + \tan(x)| + C.$$

Strategy 0.12.7 (Trigonometric Integration, Case III)**.** Consider the case that $n \geq 2$ is an even integer. Explicitly, assume that $n = 2k$ for some positive integer k, from which it follows that

$$\int \tan^m(x)\sec^n(x)\,dx = \int \tan^m(x)\sec^{2k}(x)\,dx = \int \tan^m(x)[\sec^2(x)]^{k-1}(\sec^2(x)\,dx).$$

Considering that $\sec^2(x) = 1 + \tan^2(x)$ and $\frac{d}{dx}\tan(x) = \sec^2(x)$, letting $u = \tan(x)$ yields that

$$\int \tan^m(x)\sec^n(x)\,dx = \int \tan^m(x)(1 + \tan^2(x))^{k-1}(\sec^2(x)\,dx) = \int u^m(1 + u^2)^{k-1}\,du.$$

Expanding the polynomial $(1 + u^2)^{k-1}$ and using the Power Rule, we can compute the integral.

Exercise 0.12.8. Compute the indefinite integral of $\tan^2(x)\sec^2(x)$.

Exercise 0.12.9. Compute the indefinite integral of $\tan^5(x)\sec^4(x)$.

Strategy 0.12.10 (Trigonometric Integration, Case IV)**.** Consider the case that $m \geq 1$ is odd and $n \geq 1$. Explicitly, assume that $m = 2\ell + 1$ for some positive integer ℓ, from which it follows that

$$\int \tan^m(x)\sec^n(x)\,dx = \int \tan^{2\ell+1}(x)\sec^n(x)\,dx = \int [\tan^2(x)]^\ell \sec^{n-1}(x)(\sec(x)\tan(x)\,dx).$$

Considering that $\tan^2(x) = \sec^2(x) - 1$ and $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$, letting $v = \sec(x)$ yields that

$$\int \tan^m(x)\sec^n(x)\,dx = \int [\sec^2(x) - 1]^\ell \sec^{n-1}(x)(\sec(x)\tan(x)\,dx) = \int (v^2 - 1)^\ell v^{n-1}\,dv.$$

Expanding the polynomial $(v^2 - 1)^{\ell-1}$ and using the Power Rule, we can compute the integral.

Exercise 0.12.11. Compute the indefinite integral of $\tan(x)\sec^2(x)$.

Exercise 0.12.12. Compute the indefinite integral of $\tan^3(x)\sec^3(x)$.

Unfortunately, it is difficult to compute the indefinite integral of the function $\tan^m(x)\sec^n(x)$ when $m \geq 2$ is an even integer and $n \geq 1$ is an odd integer; however, in this case, it is possible to use integration by parts and the Pythagorean Identity to transform the integrand into one that falls into either Trigonometric Integration, Case III or Trigonometric Integration, Case IV as follows.

Example 0.12.13. Consider the trigonometric function $\tan^2(x)\sec(x)$. Observe that if $u = \tan(x)$ and $dv = \sec(x)\tan(x)\,dx$, then by the method of Integration by Parts Formula, we have that

$$\int \tan^2(x)\sec(x)\,dx = \sec(x)\tan(x) - \int \sec^3(x)\,dx.$$

We are now in a position to compute the indefinite integral by evaluating the indefinite integral of $\sec^3(x)$. By the Pythagorean Identity $1 + \tan^2(x) = \sec^2(x)$, we have that

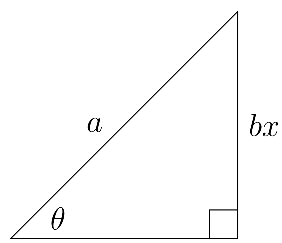
$$\int \sec^3(x)\,dx = \int \sec(x)[\tan^2(x) + 1]\,dx = \int \tan^2(x)\sec(x)\,dx + \int \sec(x)\,dx.$$

By plugging this back into our above displayed equation and rearranging, it follows that

$$\int \tan^2(x)\sec(x) = \frac{1}{2}\left[\sec(x)\tan(x) - \int \sec(x)\,dx\right] = \frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln|\sec(x) + \tan(x)| + C.$$

0.13 Trigonometric Substitution

Beyond their extensive applications in geometry and physics, the trigonometric functions yield a very powerful substitution method for integration. Consider the following right triangle.



By the Pythagorean Theorem, the side adjacent to the interior angle θ has length $\sqrt{a^2 - b^2x^2}$ so that $a\cos(\theta) = \sqrt{a^2 - b^2x^2}$. Observe that $bx = a\sin(\theta)$ so that $b\,dx = a\cos(\theta)\,d\theta$, and we have that

$$\begin{aligned}
\int \sqrt{a^2 - b^2x^2}\,dx &= \int a\cos(\theta)\left(\frac{a}{b}\cos(\theta)\,d\theta\right) \\
&= \frac{a^2}{b}\int \cos^2(\theta)\,d\theta \\
&= \frac{a^2}{2b}\int [1 + \cos(2\theta)]\,d\theta && \text{(power-reduction formula)} \\
&= \frac{a^2}{2b}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{a^2}{2b}[\theta + \sin(\theta)\cos(\theta)] + C && \text{(double-angle formula)} \\
&= \frac{a^2}{2b}\left[\arcsin\left(\frac{bx}{a}\right) + \frac{bx}{a^2}\sqrt{a^2 - b^2x^2}\right] + C,
\end{aligned}$$

where the last equality comes from the substitution $bx = \sin(\theta)$ and the above triangle.

Strategy 0.13.1 (Trigonometric Substitution, Case I)**.** Given a function $f(x)$ that can be written as $g(x)\sqrt{a^2 - b^2x^2}$ for some nonzero real numbers a and b and some function $g(x)$, we may attempt to compute $\int f(x)\,dx$ by making the substitution $bx = a\sin(\theta)$ so that $b\,dx = a\cos(\theta)\,d\theta$.

Example 0.13.2. Use a trigonometric substitution to compute the indefinite integral of $x^2\sqrt{1-x^2}$.

Solution. Considering that this function has a factor of $\sqrt{1-x^2}$, we may make the trigonometric substitution $x = \sin(\theta)$ so that $dx = \cos(\theta)\,d\theta$. Observe that $x^2 = \sin^2(\theta)$ so that by the Pythagorean Identity, we have that $\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta)$. Consequently, we find that

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2(\theta)\cos(\theta)(\cos(\theta)\,d\theta) = \int \sin^2(\theta)\cos^2(\theta)\,d\theta.$$

By Example 0.12.5 above and the double-angle formulas, we have that

$$\begin{aligned}\int \sin^2(\theta)\cos^2(\theta)\,d\theta &= \frac{1}{8}\left[\theta - \frac{1}{4}\sin(4\theta)\right] + C \\ &= \frac{1}{8}\left[\theta - \frac{1}{2}\sin(2\theta)\cos(2\theta)\right] + C \\ &= \frac{1}{8}(\theta - \sin(\theta)\cos(\theta)[\cos^2(\theta) - \sin^2(\theta)]) + C \\ &= \frac{1}{8}[\theta - \sin(\theta)\cos^3(\theta) + \sin^3(\theta)\cos(\theta)] + C.\end{aligned}$$

Using the substitution $x = \sin(\theta)$ and the fact that $\sqrt{1-x^2} = \cos(\theta)$, we conclude that

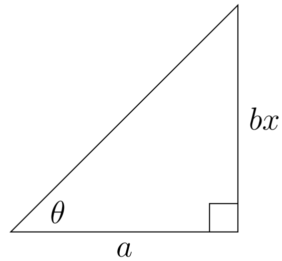
$$\int x^2\sqrt{1-x^2}\,dx = \frac{1}{8}\left[\arcsin(x) - x(1-x^2)^{3/2} + x^3\sqrt{1-x^2}\right] + C. \qquad \diamond$$

Exercise 0.13.3. Use a trigonometric substitution to compute the indefinite integral of $\frac{x}{\sqrt{1-x^2}}$.

Exercise 0.13.4. Use a trigonometric substitution to compute the indefinite integral of $x^5\sqrt{1-9x^2}$.

Exercise 0.13.5. Use a trigonometric substitution to compute the indefinite integral of $\frac{x^2}{\sqrt{9-x^2}}$.

Certainly, it is possible to consider other possibilities for our initial right triangle. Explicitly, suppose that the altitude and base of a right triangle are given as follows.



By the Pythagorean Theorem, the hypotenuse of the above right triangle has length $\sqrt{a^2+b^2x^2}$. Observe that $bx = a\tan(\theta)$ so that $b\,dx = a\sec^2(\theta)\,d\theta$. Consequently, we have that

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2+b^2x^2}} &= \int \frac{\frac{a}{b}\sec^2(\theta)\,d\theta}{\sqrt{a^2+a^2\tan^2(\theta)}} \\
&= \frac{a}{b}\int \frac{\sec^2(\theta)\,d\theta}{\sqrt{a^2(1+\tan^2(\theta))}} \\
&= \frac{a}{b}\int \frac{\sec^2(\theta)\,d\theta}{\sqrt{a^2\sec^2(\theta)}} && \text{(Pythagorean Identity)} \\
&= \frac{1}{b}\int \sec(\theta)\,d\theta && (a > 0 \text{ and } \sec(\theta) > 0) \\
&= \frac{1}{b}\ln|\sec(\theta)+\tan(\theta)| + C \\
&= \frac{1}{b}\ln\left|\frac{\sqrt{a^2+b^2x^2}+bx}{a}\right| + C,
\end{aligned}$$

where the last equality comes from the substitution $bx = a\tan(\theta)$ and the above triangle.

Strategy 0.13.6 (Trigonometric Substitution, Case II)**.** Given a function $f(x)$ that can be written as $g(x)\sqrt{a^2+b^2x^2}$ for some nonzero real numbers a and b and some function $g(x)$, we may attempt to compute $\int f(x)\,dx$ by making the substitution $bx = a\tan(\theta)$ so that $b\,dx = a\sec^2(\theta)\,d\theta$.

Example 0.13.7. Use a trigonometric substitution to compute the indefinite integral of $x^3\sqrt{1+x^2}$.

Solution. Considering that this function has a factor of $\sqrt{1+x^2}$, we may make the trigonometric substitution $x = \tan(\theta)$ with $dx = \sec^2(\theta)\,d\theta$. Observe that $x^2 = \tan^2(\theta)$ so that by the Pythagorean Identity, we have that $\sqrt{1+x^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = \sec(\theta)$. Consequently, we find that

$$\int x^3\sqrt{1+x^2}\,dx = \int \tan^3(\theta)\sec(\theta)(\sec^2(\theta)\,d\theta) = \int \tan^3(\theta)\sec^3(\theta)\,d\theta.$$

We are now in a position to evaluate a trigonometric integral. By the technique outlined in Trigonometric Integration, Case IV, we may borrow a factor of $\tan(\theta)$ and a factor of $\sec(\theta)$ and use the Pythagorean Identity $\tan^2(\theta) = \sec^2(\theta) - 1$ to simplify the integrand $\tan^3(\theta)\sec^3(\theta)\,d\theta$ as follows.

$$\int \tan^3(\theta)\sec^3(\theta)\,d\theta = \int (\sec^2(\theta)-1)\sec^2(\theta)(\sec(\theta)\tan(\theta)\,d\theta)$$

We now employ the substitution $u = \sec(\theta)$ with $du = \sec(\theta)\tan(\theta)\,d\theta$ to obtain the following.

$$\int (\sec^2(\theta)-1)\sec^2(\theta)(\sec(\theta)\tan(\theta)\,d\theta) = \int (u^2-1)u^2\,du = \int (u^4-u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C.$$

Considering that $u = \sec(\theta) = \sqrt{1 + x^2}$, it follows that

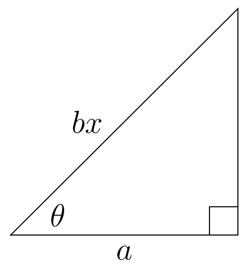
$$\int x^3 \sqrt{1+x^2}\, dx = \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C. \qquad \diamond$$

Exercise 0.13.8. Use a trigonometric substitution to compute the indefinite integral of $(x^2+1)^{-3/2}$.

Exercise 0.13.9. Use a trigonometric substitution to compute the indefinite integral of $x^2(x^2+9)^{3/2}$.

Exercise 0.13.10. Use a trigonometric substitution to compute the indefinite integral of $x^5\sqrt{4+x^2}$.

Last, consider the following right triangle in which the base and hypotenuse are given.



By the Pythagorean Theorem, the side opposite the interior angle θ has length $\sqrt{b^2x^2 - a^2}$. Observe that $bx = a\sec(\theta)$ so that $b\,dx = a\sec(\theta)\tan(\theta)\,d\theta$. Consequently, we have that

$$\begin{aligned}
\int \frac{dx}{b^2x^2 - a^2} &= \int \frac{\frac{a}{b}\sec(\theta)\tan(\theta)\,d\theta}{a^2\tan^2(\theta)} \\
&= \frac{1}{ab}\int \frac{\sec(\theta)\,d\theta}{\tan(\theta)} \\
&= \frac{1}{ab}\int \csc(\theta)\,d\theta \\
&= -\frac{1}{ab}\ln|\csc(\theta) + \cot(\theta)| + C \\
&= -\frac{1}{ab}\ln\left|\frac{bx + a}{\sqrt{b^2x^2 - a^2}}\right| + C,
\end{aligned}$$

where the last equality comes from the substitution $bx = a\sec(\theta)$ and the above triangle.

Strategy 0.13.11 (Trigonometric Substitution, Case III)**.** Given a function $f(x)$ that can be written as $g(x)\sqrt{b^2x^2 - a^2}$ for some nonzero real numbers a and b and some function $g(x)$, we may attempt to compute $\int f(x)\,dx$ via the substitution $bx = a\sec(\theta)$ so that $b\,dx = a\sec(\theta)\tan(\theta)\,d\theta$.

Example 0.13.12. Use a trigonometric substitution to compute the indefinite integral of $x^3\sqrt{x^2 - 1}$.

Solution. Considering that this function has a factor of $\sqrt{x^2-1}$, we may make the substitution $x = \sec(\theta)$ so that $dx = \sec(\theta)\tan(\theta)\,d\theta$. Observe that $x^2 = \sec^2(\theta)$ so that by the Pythagorean Identity, we have that $\sqrt{x^2-1} = \sqrt{\sec^2(\theta)-1} = \sqrt{\tan^2(\theta)} = \tan(\theta)$. Consequently, we find that

$$\int x^3\sqrt{x^2-1}\,dx = \int \sec^3(\theta)\tan(\theta)(\sec(\theta)\tan(\theta)\,d\theta) = \int \tan^2(\theta)\sec^4(\theta)\,d\theta.$$

Observe that we may use the substitution $u = \tan(\theta)$ with $du = \sec^2(\theta)\,d\theta$ to obtain

$$\begin{aligned}
\int \tan^2(\theta)\sec^4(\theta)\,d\theta &= \int \tan^2(\theta)\sec^2(\theta)(\sec^2(\theta)\,d\theta) \\
&= \int \tan^2(\theta)(1+\tan^2(\theta))(\sec^2(\theta)\,d\theta) && \text{(Pythagorean Identity)} \\
&= \int u^2(1+u^2)\,du \\
&= \int (u^2+u^4)\,du \\
&= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C.
\end{aligned}$$

Considering that $u = \tan(\theta) = \sqrt{x^2-1}$, it follows that

$$\int x^3\sqrt{x^2-1}\,dx = \frac{1}{3}(x^2-1)^{3/2} + \frac{1}{5}(x^2-1)^{5/2} + C. \qquad \diamond$$

Exercise 0.13.13. Use a trigonometric substitution to compute $\int (x^2-4)^{-3/2}\,dx$.

Exercise 0.13.14. Use a trigonometric substitution to compute $\int \sqrt{4x^2-9}\,dx$.

Exercise 0.13.15. Use a trigonometric substitution to compute $\int x^5\sqrt{x^2-16}\,dx$.

0.14 Partial Fraction Decomposition

We have thus far discussed several satisfactory techniques for integrating power functions, algebraic functions, exponential functions, logarithmic functions, trigonometric functions, and their products; however, we have not yet uniformly dealt with the problem of integrating **rational functions**. By definition, a rational function is a quotient of two polynomial expressions, e.g., the rational functions

$$\frac{1}{x^2+2x} \text{ and } \frac{x-2}{x-5} \text{ and } \frac{x^3-1}{x^2+1}.$$

We say that a rational function is **proper** if and only if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. Of the displayed rational functions above, only the first is a proper rational function. By performing **polynomial long division**, we may convert any **improper** rational function into a linear combination of proper rational functions. Explicitly, we have that $x - 2 = (x - 5) + 3$ so that dividing each side by $x - 5$ yields that

$$\frac{x-2}{x-5} = 1 + \frac{3}{x-2}.$$

We may subsequently compute the antiderivative of this rational function by elementary methods.

$$\int \frac{x-2}{x-5}\,dx = \int \left(1 + \frac{3}{x-2}\right) dx = \int 1\,dx + \int \frac{3}{x-2}\,dx = x + 3\ln|x-2| + C$$

Likewise, by polynomial long division, we find that $x^3 - 1 = x(x^2+1) - (x+1)$ so that the improper rational function can be written as the following linear combination of proper rational functions.

$$\frac{x^3-1}{x^2+1} = x - \frac{x+1}{x^2+1} = x - \frac{x}{x^2+1} - \frac{1}{x^2+1}$$

Once again, the antiderivative of this rational function can be found with relative ease.

$$\int \frac{x^3-1}{x^2+1}\,dx = \int x\,dx - \int \frac{x}{x^2+1}\,dx - \frac{1}{x^2+1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) - \arctan(x) + C$$

Unfortunately, the antiderivative of the proper rational function $(x^2 + 2x)^{-1}$ cannot be obtained by any technique we have discussed so far; however, it is possible to integrate this function by noticing (quite cleverly) that it can be written as a difference of proper rational functions as follows.

$$\int \frac{1}{x^2+2x}\,dx = \int \left(\frac{1}{2x} - \frac{1}{2(x+2)}\right) dx = \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x+2}\,dx = \frac{1}{2}(\ln|x| - \ln|x+2|) + C$$

Essentially, the content of this observation is the method of **partial fraction decomposition**.

Before we delve into the method of partial fraction decomposition, we must continue to recall some important notions from college algebra. We say that a polynomial is **irreducible** if it cannot be written as a product of two polynomials of strictly lesser degree. Consequently, a linear polynomial $ax + b$ is irreducible; it can be shown that a quadratic polynomial is irreducible if and only if it has no roots. By the Quadratic Equation, the roots of a real quadratic polynomial $ax^2 + bx + c$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that ax^2+bx+c is irreducible if and only if $b^2-4ac < 0$. We refer to the real number b^2-4ac as the **discriminant** of the quadratic: if this quantity is negative, the quadratic has only imaginary roots. One of the most useful (and nontrivial) facts about real polynomials is that the only irreducible polynomials with real coefficients are linear or quadratic. Put another way, it turns out that every real polynomial factors as a product of linear and irreducible quadratic polynomials.

Theorem 0.14.1 (Partial Fraction Decomposition Theorem)**.**

(a.) (Distinct Linear Factors) *Given any real numbers a, b, c, and d such that a and c are nonzero and $ax + b$ and $cx + d$ are distinct, there exist nonzero real numbers A and B such that*

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}.$$

(b.) (Powers of Distinct Linear Factors) *Given any real numbers a, b, c, and d such that a and c are nonzero and $ax+b$ and $cx+d$ are distinct and any pair of positive integers m and n, there exist real numbers $A_1, A_2, \dots, A_m$ and $B_1, B_2, \dots, B_n$ not all of which are zero such that*

$$\frac{1}{(ax+b)^m(cx+d)^n} = \sum_{i=1}^{m} \frac{A_i}{(ax+b)^i} + \sum_{j=1}^{n} \frac{B_j}{(cx+d)^j}.$$

(c.) (Linear and Irreducible Quadratic Factors) *Given any real numbers a, b, c, d, and e such that a and c are nonzero and $d^2 - 4ce < 0$, there exist real numbers A, B, C not all zero such that*

$$\frac{1}{(ax+b)(cx^2+dx+e)} = \frac{A}{ax+b} + \frac{Bx+C}{cx^2+dx+e}.$$

(d.) (Distinct Irreducible Quadratic Factors) *Given any real numbers a, b, c, d, e, and f such that a and d are nonzero, ax^2+bx+c and dx^2+ex+f are distinct, $b^2-4ac < 0$, and $e^2-4df < 0$, there exist real numbers A, B, C, and D not all of which are zero such that*

$$\frac{1}{(ax^2+bx+c)(dx^2+ex+f)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{dx^2+ex+f}.$$

(e.) (Powers of Distinct Irreducible Quadratic Factors) *Given any pair of positive integers m and n and any real numbers a, b, c, d, e, and f such that a and d are nonzero, $b^2 - 4ac < 0$, $e^2 - 4df < 0$, and $ax^2 + bx + c$ and $dx^2 + ex + f$ are distinct, there exist real numbers $A_1, \dots, A_m, B_1, \dots, B_m, C_1, \dots, C_n$, and $D_1, D_2, \dots, D_n$ not all of which are zero such that*

$$\frac{1}{(ax^2+bx+c)^m(dx^2+ex+f)^n} = \sum_{i=1}^{m} \frac{A_ix+B_i}{(ax^2+bx+c)^i} + \sum_{j=1}^{n} \frac{C_jx+D_j}{(dx^2+ex+f)^j}.$$

Even more, these are all of the possible cases of proper rational functions with numerator 1.

Example 0.14.2. Use the Partial Fraction Decomposition Theorem to compute $\displaystyle\int \frac{1}{x^2-5x-6}\,dx$.

Solution. Observe that $x^2 - 5x - 6 = (x-6)(x+1)$ is a factorization of $x^2 - 5x - 6$ into distinct linear factors, hence the method of partial fraction decomposition yields that

$$\frac{1}{x^2-5x-6} = \frac{A}{x-6} + \frac{B}{x+1}.$$

Clearing denominators and using the fact that $(x-6)(x+1) = x^2 - 5x - 6$, we find that

$$1 = A(x+1) + B(x-6).$$

By setting $x = 6$, we find that $1 = 7A$ so that $A = \frac{1}{7}$. By setting $x = -1$, we find that $1 = -7B$ so that $B = -\frac{1}{7}$. Consequently, the method of partial fraction decomposition reveals that

$$\frac{1}{x^2-5x-6} = \frac{\frac{1}{7}}{x-6} + \frac{-\frac{1}{7}}{x+1}.$$

We may therefore return to compute our indefinite integral with elementary techniques.

$$\int \frac{1}{x^2-5x-6}\,dx = \frac{1}{7}\int \frac{1}{x-6}\,dx - \frac{1}{7}\int \frac{1}{x+1}\,dx = \frac{1}{7}\ln|x-6| - \frac{1}{7}\ln|x+1| + C. \qquad \diamond$$

Example 0.14.3. Use the method of partial fraction decomposition to compute $\int (x^4-1)^{-1}\,dx$.

Solution. Observe that $x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$ is a factorization of x^4-1 into distinct linear and quadratic factors. Considering that $0-4(1)(1)=-4<0$, it follows that x^2+1 is irreducible. Using the method of partial fraction decomposition, it follows that

$$\frac{1}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}.$$

Clearing denominators and using the fact that $(x-1)(x+1)=x^2-1$, we find that

$$1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1).$$

Considering that this identity holds for all x, it follows that $4A=1$ by plugging in $x=1$, $-4B=1$ by plugging in $x=-1$, and $A-B-D=1$ by plugging in $x=0$. We find immediately that

$$A=\frac{1}{4},\ B=-\frac{1}{4},\ \text{and } D=A-B-1=\frac{1}{2}-1=-\frac{1}{2}.$$

Expanding the polynomial on the right in the second-to-last displayed equation, we find that

$$0x^3+1=1=(A+B+C)x^3+\text{some polynomial of degree at most two.}$$

Comparing coefficients gives that $A+B+C=0$ so that $C=0$. We conclude that

$$\frac{1}{x^4-1}=\frac{\frac{1}{4}}{x-1}-\frac{\frac{1}{4}}{x+1}-\frac{\frac{1}{2}}{x^2+1},$$

from which it follows that

$$\int\frac{1}{x^4+1}\,dx=\frac{1}{4}\int\frac{1}{x-1}\,dx-\frac{1}{4}\int\frac{1}{x+1}-\frac{1}{2}\int\frac{1}{x^2+1}\,dx$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\arctan(x)+C.$$ ◇

Caution: it is not necessarily always possible to eliminate variables by plugging in carefully chosen values $x=a$ when implementing the method of partial fraction decomposition. Ultimately, it is in fact best to use the **method of undetermined coefficients**, as outlined in our next example.

Example 0.14.4. Use the Partial Fraction Decomposition Theorem to compute $\displaystyle\int\frac{2x+1}{x^4+2x^2+1}\,dx$.

Solution. Observe that $x^4+4x^2+3=(x^2+1)(x^2+3)$ is a factorization of x^4+4x^2+3 into distinct irreducible factors. Using the method of partial fraction decomposition, we have that

$$\frac{2x+1}{(x^2+1)(x^2+3)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}.$$

Clearing denominators, we find that

$$2x+1=(Ax+B)(x^2+3)+(Cx+D)(x^2+1).$$

Considering that x^2+1 and x^2+3 are irreducible, we cannot eliminate either of these quadratic factors by substituting $x=a$ for any real number a. Consequently, we must compare coefficients. Expanding the right-hand side in the second-to-last displayed equation, we find that

$$2x+1=(A+C)x^3+(B+D)x^2+(3A+C)x+3B+D,$$

from which we obtain the following linear system of equations.

$$\begin{aligned} A+C&=0 & \qquad 3A+C&=2 \\ B+D&=0 & \qquad 3B+D&=1 \end{aligned}$$

We have therefore that $A=-C$ and $B=-D$ so that $2=-3C+C=-2C$ and $1=-3D+D=-2D$. We conclude that $A=1$, $B=\frac{1}{2}$, $C=-1$, and $D=-\frac{1}{2}$, from which it follows that

$$\begin{aligned} \int \frac{2x+1}{x^4+4x^2+3}\,dx &= \int \left(\frac{x+\frac{1}{2}}{x^2+1}-\frac{x+\frac{1}{2}}{x^2+3}\right)dx \\ &= \frac{1}{2}\int \frac{2x+1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x+1}{x^2+3}\,dx \\ &= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+3}\,dx - \frac{1}{2}\int \frac{1}{x^2+3}\,dx \\ &= \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\arctan(x) - \frac{1}{2}\ln|x^2+3| - \frac{1}{2\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C, \end{aligned}$$

where the last integral is determined by $x^2+3=3\left[\left(\frac{x}{\sqrt{3}}\right)^2+1\right]$ and the substitution $u=\frac{x}{\sqrt{3}}$. ◇

Example 0.14.5. Use the Partial Fraction Decomposition Theorem to compute $\int \frac{1}{x^2-1}\,dx$.

Exercise 0.14.6. Use the method of partial fraction decomposition to compute $\int \frac{2x+3}{x^3-2x^2+4x-8}\,dx$.

Observe that the method of partial fraction decomposition applies to proper rational functions; however, by polynomial long division, every rational function induces a proper rational function.

Example 0.14.7. Use polynomial long division to express the following rational function as the sum of a polynomial and a proper rational function; then, compute its indefinite integrals.

$$f(x)=\frac{x^3+1}{x^2+x+1}$$

Solution. We proceed by polynomial long division. Our task is to sequentially eliminate the largest power of x in each polynomial that appears as the **dividend** in the long division.

1.) Our dividend is x^3+1, and our **divisor** is x^2+x+1. Observe that

$$(x^3+1)-x(x^2+x+1)=(x^3+1)-(x^3+x^2+x)=-x^2-x+1.$$

2.) Our dividend is now $-x^2 - x + 1$, and our divisor is $x^2 + x + 1$. Observe that

$$(-x^2 - x + 1) - (-1)(x^2 + x + 1) = (-x^2 - x + 1) + (x^2 + x + 1) = 2.$$

3.) Our dividend of 2 has lesser degree than $x^2 + x + 1$, so the division terminates.

$$\begin{array}{r|l} & \phantom{-x^3 - x^2 - {}} x - 1 \\ \hline x^2 + x + 1 & x^3 + 1 \\ & -x^3 - x^2 - x \\ \hline & - x^2 - x + 1 \\ & \phantom{-x^3 - {}} x^2 + x + 1 \\ \hline & \phantom{-x^3 - x^2 - x + {}} 2 \end{array}$$

Ultimately, we find that $x^3 + 1 = (x - 1)(x^2 + x + 1) + 2$ so that

$$\frac{x^3 + 1}{x^2 + x + 1} = x - 1 + \frac{2}{x^2 + x + 1}.$$

Considering that $1^2 - 4(1)(1) = -3 < 0$, it follows that $x^2 + x + 1$ is an irreducible quadratic polynomial, hence the method of partial fraction decomposition fails to improve the situation here; rather, we may revert to the method of **completing the square** to find that

$$x^2 + x + 1 = x^2 + x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

By setting $u = x + \frac{1}{2}$, we find that $du = dx$ so that

$$\int \frac{2}{x^2 + x + 1}\, dx = 2 \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}\, dx = 2 \int \frac{1}{u^2 + \frac{3}{4}}\, du = \frac{8}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}} u\right)^2 + 1}\, du.$$

One can perform a substitution $t = \frac{2}{\sqrt{3}} u$ with $dt = \frac{2}{\sqrt{3}}\, du$ or simply recognize this integral as

$$\frac{8}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}} u\right)^2 + 1}\, du = \frac{4}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}} u\right) + C = \frac{4}{\sqrt{3}} \arctan\left(\frac{2x + 1}{\sqrt{3}}\right) + C.$$

Ultimately, we conclude that the function has the following general antiderivative.

$$\int \frac{x^3 + 1}{x^2 + x + 1}\, dx = \int \left(x - 1 + \frac{2}{x^2 + x + 1}\right) dx = \frac{1}{2} x^2 - x + \frac{4}{\sqrt{3}} \arctan\left(\frac{2x + 1}{\sqrt{3}}\right) + C \qquad \diamond$$

Example 0.14.8. Use polynomial long division to express the following rational functions as the sum of a polynomial and a proper rational function; then, compute their indefinite integrals.

(a.) $\dfrac{x^3 + 1}{x^2 + x + 1}$ (b.) $\dfrac{x^4 - x^2 + 1}{x^2 - 1}$ (c.) $\dfrac{x^5 - 4x^4 + 9x^2 - 6}{x^3 + x^2 - x - 1}$

0.15 Improper Integration

Our interest in integrals so far has been to find the net area bounded by the curve $f(x)$. Because of this, we have restricted ourselves to closed and bounded intervals of the form $[a, b]$. Often, we are interested in how a mathematical model behaves in the long-run, i.e., as x grows arbitrarily large (or approaches $\pm\infty$). Under this framework, we develop the concept of the improper integral.

Given a function $f(x)$ that is integrable over the closed region $[a, b]$ for every real number $b > a$, the **improper integral** of $f(x)$ over the interval $[a, \infty)$ is defined (if it exists) as

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

By the Fundamental Theorem of Calculus, Part I, for any antiderivative $F(x)$ of $f(x)$, we have that

$$\lim_{b\to\infty} \int_a^b f(x)\,dx = \lim_{b\to\infty} [F(b) - F(a)].$$

One can analogously define the improper integral of $f(x)$ over the interval $(-\infty, b]$ as

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

whenever $f(x)$ is integrable over the closed and bounded interval $[a, b]$ for every real numbers $a < b$. Even more, the doubly improper integral of $f(x)$ over $(-\infty, \infty)$ is defined as

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{b\to\infty} \left(\lim_{a\to-\infty} \int_a^b f(x)\,dx \right) = \lim_{a\to-\infty} \left(\lim_{b\to\infty} \int_a^b f(x)\,dx \right)$$

whenever $f(x)$ is integral over the closed and bounded interval $[a, b]$ for all real numbers a and b.

Exercise 0.15.1. Compute the improper integral $\int_1^\infty x^{-2}\,dx$.

Exercise 0.15.2. Compute the improper integral $\int_{-\infty}^1 e^x\,dx$.

Exercise 0.15.3. Compute the improper integral $\int_0^\infty xe^{-x}\,dx$.

Exercise 0.15.4. Compute the improper integral $\int_{-\infty}^\infty (1 + x^2)^{-1}\,dx$.

Exercise 0.15.5. Compute the improper integral $\int_{-\infty}^\infty xe^{-x^2}\,dx$.

Each of the above functions admits horizontal asymptotes, hence the improper integrals we computed were all finite, and the ends of our computations justified the means.

One can also consider the improper integral of a function with a vertical asymptote. Given that $f(x)$ is continuous on the half-open interval $[a, b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$, we have that

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx = \lim_{t\to b^-} [F(t) - F(a)]$$

for any antiderivative $F(x)$ of $f(x)$ (if this limit exists). One can analogously define the improper integral of $f(x)$ over the half-open interval $(a, b]$ whenever $\lim_{x\to a^+} f(x) = \pm\infty$ (provided it exists) as

$$\int_a^b f(x)\,dx = \lim_{u\to a^+} \int_u^b f(x)\,dx = \lim_{u\to a^+} [F(b) - F(u)].$$

Even if the integrand $f(x)$ is unbounded as $x > a$ approaches a and as $x < b$ approaches b, it is still possible to define the doubly improper integral of $f(x)$ over (a, b) as

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \left(\lim_{u \to a^+} \int_u^t f(x)\,dx \right) = \lim_{u \to a^+} \left(\lim_{t \to b^-} \int_u^t f(x)\,dx \right)$$

provided that $f(x)$ is integrable over the closed interval $[u, t]$ for all real numbers $a < u < t < b$.

Exercise 0.15.6. Compute the improper integral $\int_0^1 (x-1)^{-1}\,dx$.

Exercise 0.15.7. Compute the improper integral $\int_0^1 x^{-1/2}\,dx$.

Exercise 0.15.8. Compute the improper integral $\int_{-1}^1 x^{-2/3}\,dx$.

Conventionally, we say that an improper integral **converges** whenever the limit of definition exists, and we say that it **diverges** if the limit does not exist. Even if we cannot explicitly compute an improper integral, the **Comparison Theorem** allows us to say whether it converges or diverges.

Theorem 0.15.9 (Comparison Theorem for Improper Integrals)**.** *Consider any pair of continuous functions $f(x)$ and $g(x)$ such that $f(x) \geq g(x) \geq 0$ for all real numbers $x \geq a$.*

(a.) *If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.*

(b.) *If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.*

One can make analogous statements for the improper integrals $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^b g(x)\,dx$, doubly improper integrals, and improper integrals of a function with a vertical asymptote.

Exercise 0.15.10. Determine if the improper integral $\int_0^\infty xe^x\,dx$ converges.

Exercise 0.15.11. Determine if the improper integral $\int_0^\infty x^{-2}\sin^2(x)\,dx$ converges.

Chapter 1

First Order Differential Equations

Early in our mathematics education, we are introduced in Calculus I to the notion of the derivative of a function of one real variable as the limit of a difference quotient. Explicitly, if f is any function of a real variable x, then the derivative of f with respect to x is defined as the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for all real numbers x such that the above limit exists. Essentially, the study of first order differential equations seeks to understand how an equation relating a function and its derivative determines the expression of the function (e.g., as polynomial, exponential, logarithmic, trigonometric, etc.). We will come to find that this topic enjoys wide applications in economics, engineering, and physics.

1.1 Mathematical Modeling and Slope Fields

Generally speaking, a **mathematical model** is typically a system of functions or equations that describes the mathematics of some real-world phenomena or physical process.

Example 1.1.1. By Newton's Second Law of Motion, the force F acting on an object of mass m with acceleration a is given by $F = ma$, i.e., force equals mass times acceleration.

Example 1.1.2. Given any real function $f(x)$ that is integrable on a closed and bounded interval $[a, b]$, the average value of $f(x)$ on $[a, b]$ is given by

$$\frac{1}{b-a} \int_a^b f(x)\,dx.$$

Explicitly, if $f(x)$ measures the amount of acetaminophen in a person's bloodstream x minutes after taking one dose, the average amount of acetaminophen in the bloodstream after a minutes is

$$\frac{1}{a} \int_0^a f(x)\,dx.$$

Equations relating a function f and its derivatives f', f'', etc., are called **differential equations**. We will find in this course widespread applications of differential equations to mathematical models.

Example 1.1.3. Objects in free-fall near the Earth's surface experience the force of gravity acting to pull them downward; however, this motion also results in an opposing force called the **drag** force that pushes the object upward. Out of desire for simplicity, we may assume that the drag force δ is directly proportional to the velocity v at which the object moves, hence there exists a positive real number γ called the **drag coefficient** (or the **constant of proportionality** in generality) such that the drag force is given by $\delta = \gamma v$. Consequently, if we denote by m the mass of an object and denote by g the acceleration of an object due to gravity as it falls toward the surface of the Earth, the net force acting on the object in free-fall is governed by the mathematical model

$$F = mg - \gamma v.$$

Explicitly, according to Newton's Second Law of Motion, gravity acts on the object with a force of mg while drag acts on the object with an opposing force of γv. Considering that force is equal to mass times acceleration (once again by Newton's Second Law of Motion) and acceleration is the derivative of velocity, if we view v as a function of time t, we obtain a differential equation

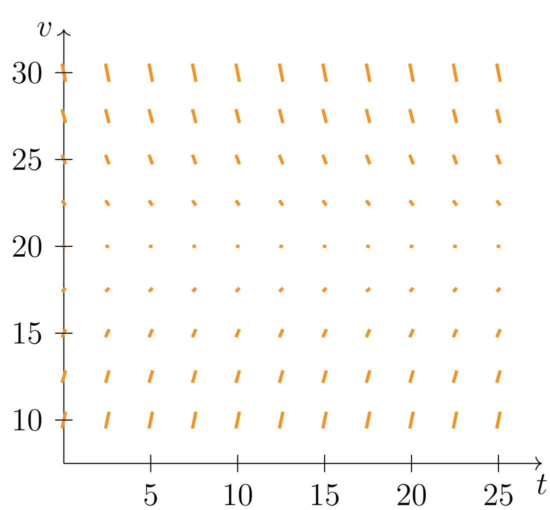
$$m\frac{dv}{dt} = mg - \gamma v.$$

Observe that the constants m, g, and γ are completely determined, hence the only unknown quantities in the above displayed equation are the velocity function v and its derivative dv/dt; however, the derivative dv/dt of the velocity with respect to time is uniquely determined by the function v, so in some sense, the only unknown quantity in the above differential equation is the velocity.

Example 1.1.4. Consider an object of mass $m = 10$ kg in free-fall with drag coefficient $\gamma = 5$ kg/s. We will assume that acceleration due to gravity is 9.8 m/s^2. By Example 1.1.3, the net force acting on the object as it falls toward the surface of the Earth is governed by the differential equation

$$10\frac{dv}{dt} = (10)(9.8) - 5v.$$

We may view this equation as an expression of the acceleration dv/dt as a function of the velocity v as suggested in the previous example. Concretely, if $v = 30$ m/s, then $dv/dt = -5.2$ m/s^2. Likewise, if $v = 10$ m/s, then $dv/dt = 4.8$ m/s^2. Each possible value of the velocity v gives rise to a value of the acceleration dv/dt. Considering that dv/dt is the slope of the line tangent to v at time t, we can plot this information graphically in the tv-plane by placing a line segment of slope dv/dt for each value of v. We refer to this plot as the **slope field** of the differential equation.



We say that a real function f that satisfies a differential equation is a **solution** of the attendant differential equation. Explicitly, in Example 1.1.4, the constant function $v(t) = 19.6$ satisfies that

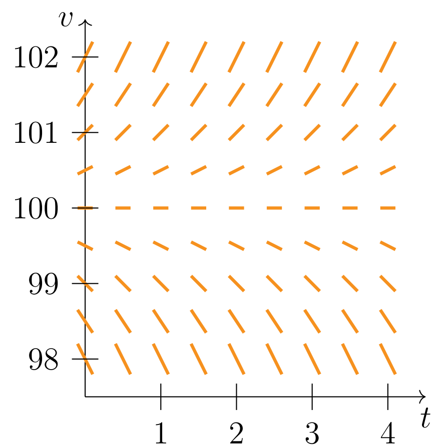
$$\frac{dv}{dt} = 0 = 9.8 - \frac{1}{2}v(t),$$

hence this function is a solution of the differential equation $10dv/dt = 980 - 5v$. Constant solutions of differential equations are called **equilibrium solutions**. Consequently, in order to find equilibrium solutions of a differential equation, it suffices to set all derivatives equal to zero.

Example 1.1.5. Consider a population of bacteria that grows according to the differential equation

$$\frac{dp}{dt} = 1.01p - 101,$$

where $p(t)$ measures the population of bacteria (in billions) t months after our initial observation. By setting $dp/dt = 0$ and solving for p, we find that $p(t) = 100$ achieves equilibrium, hence there is only one equilibrium solution of the above differential equation. Put another way, if we begin with 100 billion bacteria, then the population of bacteria remains stable over time. By sketching the slope field of this differential equation, we can predict the behavior of a population of bacteria.



Beginning with more than 100 billion bacteria, the population grows with time, as the above diagram corroborates; with fewer than 100 million bacteria, the population decays.

Example 1.1.6. Generally, the most simple differential equations satisfy that

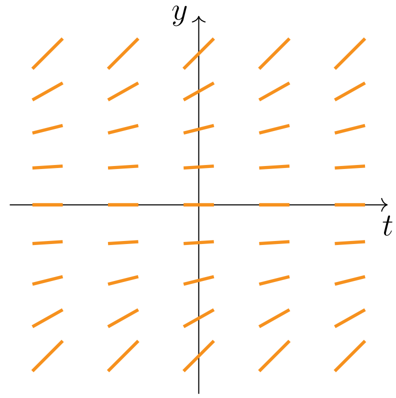
$$\frac{dy}{dt} = ay + b$$

for some real numbers a and b since the slope dy/dt of the equation $y = f(t)$ is linear with respect to time. We refer to the real number a as the **rate constant**. Crucially, a positive rate constant indicates growth because the slope dy/dt will tend to ∞ as y tends to ∞. Conversely, a negative rate constant indicates decay because dy/dt will tend to $-\infty$ as y tends to ∞. We refer to the real number b as the **predation rate**; the equilibrium solution is $y_0(t) = -b/a$ if a is nonzero.

Example 1.1.7. Certainly, there are differential equations in which the slope is not linear, e.g.,

$$\frac{dy}{dt} = y^2.$$

Observe that for this model, the slope dy/dt of the function $y = f(t)$ varies quadratically with the value of y. Below is a diagram of the slope field for this differential equation.



1.2 Elementary Differential Equations and Their Solutions

We have noticed thus far that differential equations arise naturally in the context of observable physical phenomena. Even more, we have seen that it is possible to garner some understanding of the behavior of a differential equation by sketching the slope fields; however, we have not yet developed any hard-and-fast rules to obtain solutions of differential equations. We address this as follows. Given any real numbers a and b, consider any differential equation of the form

$$\frac{dy}{dt} = ay + b.$$

Observe that if a is nonzero and y is not equal to $-b/a$, then we have that

$$\frac{dy/dt}{y + b/a} = a$$

by factoring a from the right-hand side and dividing by the resulting factor of $y + b/a$. Considering this identity in terms of the differentials dy and dt and integrating both sides, we find that

$$\ln|y + b/a| = \int \frac{y'}{y + b/a}\,dt = \int a\,dt = at + C$$

or $y + b/a = e^{at+C} = e^C e^{at} = ce^{at}$ for some real number C. Last, we obtain the solution

$$y(t) = e^C e^{at} - b/a.$$

We refer to this exponential function as the **general solution** of the differential equation

$$\frac{dy}{dt} = ay + b.$$

Crucially, if we impose some **initial condition** $y(0) = y_0$ for some real number y_0, then the result is that $C = \ln|y_0 + b/a|$ or $e^C = |y_0 + b/a|$, hence we obtain a **particular solution** in this case. Generally, the problem of determining the constant C such that $y(t) = e^C e^{at} - b/a$ is a particular solution to the above differential equation is referred to as an **initial value problem**.

Example 1.2.1. Consider an object of mass $m = 10$ kg in free-fall with drag coefficient $\gamma = 5$ kg/s. We will assume that acceleration due to gravity is 9.8 m/s^2. By Example 1.1.4, the net force acting on the object as it falls toward the surface of the Earth is governed by the differential equation

$$\frac{dv}{dt} = 9.8 - \frac{1}{2}v.$$

By factoring $-1/2$ from the right-hand side of this equation and dividing by the resulting factor of $v - 19.6$, we obtain an equation that can be easily integrated using common rules from Calculus I

$$\ln|v - 19.6| = \int \frac{v'}{v - 19.6}\,dt = -\frac{1}{2}\int dt = -\frac{1}{2}t + C.$$

Exponentiating base e yields that $v(t) - 19.6 = e^C e^{-t/2}$, hence we obtain the general solution

$$v(t) = e^C e^{-t/2} + 19.6.$$

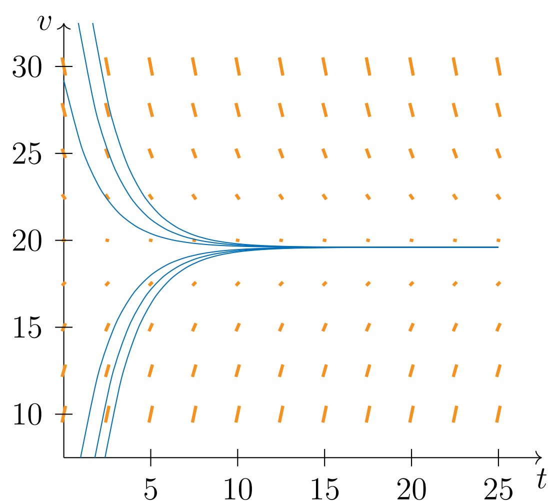
Given that the initial velocity satisfies that $v_0 = v(0) = 0$, it follows that

$$0 = v(0) = e^C + 19.6$$

so that $e^C = -19.6$, and we obtain the particular solution of the initial value problem

$$v(t) = 19.6(1 - e^{-t/2}).$$

Below is a diagram of the slope field for the differential equation (plotted in orange) as well as the **integral curves** for the general solution of the differential equation (plotted in blue).



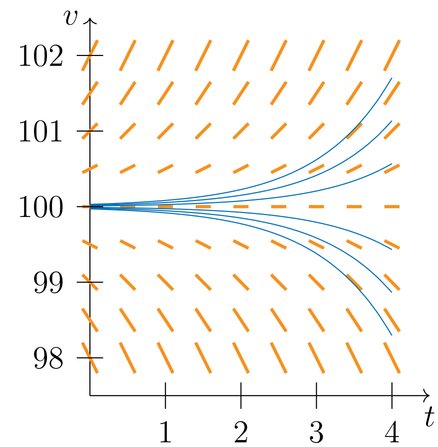
Example 1.2.2. Consider a population of bacteria that grows according to the differential equation

$$\frac{dp}{dt} = 1.01p - 101,$$

where $p(t)$ measures the population of bacteria (in billions) t months after our initial observation. Like before, if we factor 1.01 from the right-hand side of this equation, divide by the resulting factor of $p - 100$, and integrate in t, then we obtain a general solution for the differential equation

$$\ln|p - 100| = \int \frac{p'}{p - 100}\,dt = 1.01\int dt = 1.01t + C$$

or $p(t) = e^C e^{1.01t} + 100$. Given that the initial population of bacteria satisfies that $p(0) = p_0$, the population of bacteria is governed by the exponential function $p(t) = (p_0 - 100)e^{1.01t} + 100$.



1.3 Classification of Differential Equations

Until now, we have referred to differential equations only peripherally or in the context of explicit examples. We turn our attention in this section to the general classification and theory of differential equations. Certainly, there is a distinction between differential equations that depend upon a single variable (such as time) and those that depend on several variables (such as time, mass, acceleration, temperature, etc.). We refer to any differential equation in which the underlying function $y = f(t)$ depends on a single independent variable t as an **ordinary differential equation**. Consequently, an (implicit) ordinary differential equation is any differential equation of the form

$$F(t, y, y', y'', \dots, y^{(n)}) = 0$$

for some real function F, independent variable t, and real function $y = f(t)$. We have seen already that it is occasionally possible to express an ordinary differential equation explicitly as

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$$

for some real function F, independent variable t, real function $y = f(t)$, and positive integer n. Consequently, the general form of a linear ordinary differential equation is given by

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y - g(t) = 0$$

for some positive integer n and real functions $a_0(t), a_1(t), \dots, a_{n-1}(t)$, $a_n(t)$, $y = f(t)$, and $g(t)$.

Example 1.3.1. Given any real numbers a and b, we obtain an ordinary differential equation

$$y' = ay + b.$$

Last section, we constructed the general solution of an ordinary differential equation of this form.

Example 1.3.2. Given any real function $y = f(t)$, we obtain an ordinary differential equation

$$y' = y^3 + \sin(y).$$

Unfortunately, there is no closed-form expression for the general solution of this differential equation.

By analogy to ordinary derivatives and ordinary differential equations, those differential equations involving partial derivatives are **partial differential equations**. Below are some examples.

Example 1.3.3. Given any real function $u = f(t, x, y)$, the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is commonly known as the **heat equation**; it takes its name from the fact that it models how heat diffuses through a region. Often, the equation is expressed in subscript notation as $u_t = u_{xx} + u_{yy}$.

Example 1.3.4. Given any real function $u = f(t, x, y)$, the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is commonly known as the **wave equation**; it takes its name from the fact that it models the motion of travelling or standing waves. Using subscript notation, the wave equation is $u_{tt} = u_{xx} + u_{yy}$.

Every differential equation admits a derivative of highest order. Bearing this in mind, the **order** of a differential equation is simply the highest derivative that appears in the differential equation.

Example 1.3.5. Observe that the ordinary differential equation of Example 1.3.1 has order one because the highest derivative appearing in the differential equation is the first derivative. Consequently, this equation is said to be a **first order ordinary differential equation**.

Example 1.3.6. Given any real function $y = f(t)$, consider the ordinary differential equation

$$y'' = y^2.$$

Considering that the highest derivative appearing in the above equation is the second derivative, the order of this differential equation is two; it is a **second order ordinary differential equation**.

Example 1.3.7. Like ordinary differential equations, every partial differential equation admits an order: the heat equation $u_t = u_{xx} + u_{yy}$ of Example 1.3.3 and the wave equation $u_{tt} = u_{xx} + u_{yy}$ of Example 1.3.4 are both examples of **second order partial differential equations**.

Last, there is a distinction to be made regarding the manner in which the terms of a differential equation are related. Explicitly, a differential equation determined by a linear function F is referred to as a **linear** differential equation. Conversely, differential equations that are not linear are called **non-linear**. Context will clarify the distinction between linear and non-linear equations, but it is worth noting that non-linear equations are more difficult to solve in general than linear equations.

Example 1.3.8. Given any real numbers a and b and any real function $y = f(t)$, the first order differential equation $y' = ay + b$ is linear: indeed, we may view this as the differential equation

$$F(t, y, y') = y' - ay - b = 0.$$

Example 1.3.9. Given any real function $y = f(t)$, the first order differential equation

$$y' = y^3 + \sin(y)$$

is non-linear because the terms y^3 and $\sin(y)$ are non-linear functions of y.

Example 1.3.10. Given any real function $u = f(t, x, y)$, the heat equation $u_t = u_{xx} + u_{yy}$ is an example of a second order linear partial differential equation since the degree of all terms is one.

Example 1.3.11. Given any real function $u = f(t, x)$, the **Dym equation**

$$\frac{\partial u}{\partial t} = u^3 \frac{\partial^3 u}{\partial x^3}$$

is a third order non-linear partial differential equation: indeed, this equation has order three because the third partial derivative of u with respect to x appears in the equation, and it is non-linear because the term $u^3 u_{xxx}$ has degree four (as a polynomial in u, u_x, u_{xx}, and u_{xxx}).

Before we conclude this section, it is imperative that we make the following definition.

Definition 1.3.12. Given any positive integer n, any real univariate function $y = f(t)$, and any real function $F(t, y, y', y'', \dots, y^{(n)})$, a **solution** of the ordinary differential equation

$$F(t, y, y', y'', \dots, y^{(n)}) = 0$$

on the open interval (a, b) is any real univariate function $\phi(t)$ satisfying that

1.) the derivatives $\phi'(t), \phi''(t), \dots, \phi^{(n)}(t)$ exist for all real numbers $a < t < b$ and

2.) $F(t, \phi, \phi', \phi'', \dots, \phi^{(n)}) = 0$ for all real numbers $a < t < b$.

Remark 1.3.13. We will soon develop several techniques to deal with ordinary differential equations in a general setting; however, it is important to establish the following expectations.

i.) Crucially, <u>not all</u> differential equations $y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$ admit solutions: indeed, it is a matter of critical importance to determine if there exists a solution to a given differential equation. We will later refer to this predicament as the problem of **existence**.

    Even outside of the mathematical considerations that come with the question of existence, the physical ramifications therein matter significantly: for if a real-world scenario is modelled by a differential equation, then that equation ought to admit a solution. Under this identification, the validity of the model in question can be checked against the existence of a solution.

ii.) Granted that a differential equation is known to possess a solution, it is natural to wonder exactly how many solutions there are. We refer to this as the problem of **uniqueness**. We have seen thus far that any differential equation that admits a general solution must possess infinitely many solutions: indeed, it is clear that if $f(t) = F'(t)$, then $f(t)$ is the derivative of $F(t)+C$ for all real numbers C, and the Fundamental Theorem of Calculus ensures that every function whose first derivative is $f(t)$ is of the form $F(t) + C$ for some real number C. On the other hand, it is possible to impose some initial conditions in order to obtain a particular solution to a differential equation; however, the question remains as to whether there could be any further solutions. We cannot put the problem to rest until this is addressed.

iii.) Even if a differential equation admits a unique solution, it is often difficult in practice to obtain: indeed, the vast majority of differential equations do not admit solutions that can be realized in terms of elementary functions (e.g., polynomials or exponential, logarithmic, or trigonometric functions). Consequently, it is typically necessary to use numerical methods to approximate solutions to differential equation. We will deal with many types of differential equations with tidy solutions, but we will also discuss the matter of approximating solutions.

1.4 First Order Linear Equations and Integrating Factors

Consider any real bivariate function $f(t, y)$ and any first order differential equation of the form

$$\frac{dy}{dt} = f(t, y). \tag{1.4.1}$$

We will say that Equation (1.4.1) is **linear** provided that $f(t, y)$ depends linearly on the variable y. Explicitly, Equation (1.4.1) is linear if and only if there exist real univariate functions $g(t)$ and $h(t)$ such that $f(t, y) = g(t)y + h(t)$. Consequently, a **first order linear equation** in standard form is

$$\frac{dy}{dt} + p(t)y = g(t). \tag{1.4.2}$$

Unfortunately, it is typically impossible to directly solve Equation (1.4.2) by isolating the variable y and integrating both sides of the equation, as we have in previous examples; however, thanks to a technique due to Gottfried Wilhelm Leibniz, there is sometimes a way forward. Consider some real univariate function $\mu(t)$ that depends on the variable t alone. By introducing this factor into our first order linear equation and appealing to the Product Rule and Chain Rule, we find that

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \tag{1.4.3}$$

so long as the derivative of the function $\mu(t)$ with respect to t satisfies that

$$\frac{d}{dt}\mu(t) = \mu(t)p(t). \tag{1.4.4}$$

Consequently, if we assume that $\mu(t)$ is positive, then solving this differential equation yields that

$$\ln[\mu(t)] = \int p(t)\,dt.$$

Last, by composing each side of the above equation with the exponential function e^t, we obtain

$$\mu(t) = \exp\biggl(\int p(t)\,dt\biggr).$$

Crucially, we note that $\mu(t)$ is positive, as required. We refer to $\mu(t)$ as an **integrating factor**.

Algorithm 1.4.1 (Constructing an Integrating Factor)**.** Consider any first order linear equation

$$\frac{dy}{dt} + p(t)y = g(t).$$

Carry out the following steps to obtain an integrating factor $\mu(t)$ and solve the equation.

1.) Compute the antiderivative of the function $p(t)$ with respect to t.

2.) Plug the function from the last step into the exponential function to find the integrating factor

$$\mu(t) = \exp\biggl(\int p(t)\,dt\biggr).$$

3.) By construction, observe that the differential equation in Equation (1.4.3) satisfies that

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t).$$

Consequently, the solution to the differential equation at hand is given by

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt$$

so long as this antiderivative can be evaluated; otherwise, we may take

$$y(t) = \frac{1}{\mu(t)} \left(C + \int_{t_0}^{t} \mu(s)g(s)\,ds \right)$$

for some conveniently determined real number t_0 and some arbitrary real number C.

Example 1.4.2. Construct an integrating factor and solve the following differential equation.

$$\frac{dy}{dt} - y = \cos(t)$$

Solution. We note that $p(t) = -1$ and $g(t) = \cos(t)$, hence by the algorithm for Constructing an Integrating Factor, we may take $\mu(t) = e^{-t}$ since the simplest antiderivative of $p(t)$ is $-t$. Even more, the algorithm provides that the general solution of the above differential equation is given by

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt = e^t \int e^{-t}\cos(t)\,dt.$$

Computing the antiderivative in the definition of $y(t)$ amount to performing integration by parts twice, beginning with $u = e^{-t}$, $du = -e^{-t}\,dt$, $dv = \cos(t)\,dt$, and $v = \sin(t)$. Carrying this out yields

$$\int e^{-t}\cos(t)\,dt = e^{-t}\sin(t) + \int e^{-t}\sin(t)\,dt.$$

Continuing with our assignment $u = e^{-t}$, $du = -e^{-t}\,dt$, $dv = \sin(t)\,dt$, and $v = -\cos(t)$, we obtain

$$\int e^{-t}\sin(t)\,dt = -e^{-t}\cos(t) - \int e^{-t}\cos(t)\,dt.$$

Consequently, the original antiderivative can be solved according to the fact that

$$\int e^{-t}\cos(t)\,dt = e^{-t}\sin(t) - e^{-t}\cos(t) - \int e^{-t}\cos(t)\,dt.$$

By adding the common factor to each side and dividing by 2, we conclude that

$$y(t) = e^t \int e^{-t}\cos(t)\,dt = \frac{1}{2}e^t\big(e^{-t}\sin(t) - e^{-t}\cos(t) + C\big) = \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) + Ce^t. \qquad \diamond$$

Example 1.4.3. Construct an integrating factor and solve the following differential equation.

$$\frac{dy}{dt} + y = t + e^t$$

Solution. Considering that $g(t) = t + e^t$ and $p(t) = 1$, we may take $\mu(t) = e^t$ since the simplest antiderivative of $p(t)$ is t. We conclude that the general solution to this differential equation is

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt = e^{-t} \int e^t(t + e^t)\,dt = e^{-t} \int te^t\,dt + e^{-t} \int e^{2t}\,dt.$$

We can easily dispose of the second antiderivative; the first antiderivative can be solved by integration by parts with $u = t$, $du = dt$, $dv = e^t\,dt$, and $v = e^t$. Carrying this out, we find that

$$\int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t + C.$$

Consequently, we obtain the general solution for this linear first order differential equation

$$y(t) = e^{-t}(te^t - e^t + C) + e^{-t}\left(\frac{1}{2}e^{2t} + D\right) = t - 1 + \frac{1}{2}e^t + (C + D)e^{-t}. \qquad \diamond$$

Example 1.4.4. Construct an integrating factor and solve the following differential equation.

$$\frac{dy}{dt} + t^2 y = 2te^{t^2}$$

Solution. We note that $g(t) = 2te^{t^2}$ and $p(t) = t^2$. By the Power Rule, the simplest antiderivative of $p(t)$ is $\frac{1}{3}t^3$, hence the integrating factor for this equation is given by $\mu(t) = e^{t^3/3}$. Unfortunately, in this case, we cannot find an elementary antiderivative for the product function

$$\mu(t)g(t) = e^{t^3/3}(2te^{t^2}) = 2te^{t^2 + \frac{1}{3}t^3}.$$

Consequently, according to the algorithm for Constructing an Integrating Factor, the general solution to the above first order linear differential equation is given by

$$y(t) = \frac{1}{\mu(t)}\left(C + \int_{t_0}^{t} \mu(s)g(s)\,ds\right) = Ce^{-t^3/3} + e^{-t^3/3} \int_{t_0}^{t} 2se^{s^2 + \frac{1}{3}s^3}\,ds. \qquad \diamond$$

Example 1.4.5. Construct an integrating factor and solve the following differential equation.

$$t\frac{dy}{dt} + y = e^t \sin(t) \text{ for all real numbers } t > 0$$

Solution. We note that the given equation is not in standard form. By dividing each of the terms in the equation by a factor of t, we obtain a first order linear differential equation in standard form

$$\frac{dy}{dt} + \frac{1}{t}y = \frac{1}{t}e^t \sin(t).$$

Consequently, we may identify that $g(t) = t^{-1}e^t \sin(t)$ and $p(t) = t^{-1}$. Considering that $t > 0$, an antiderivative of t^{-1} is $\ln(t)$, hence we obtain an integrating factor of $\mu(t) = t$. Using integration by parts, we conclude that the general solution to the given differential equation satisfies that

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt = \frac{1}{t} \int e^t \sin(t)\,dt = \frac{1}{2t}(e^t \sin(t) - e^t \cos(t) + C). \qquad \diamond$$

1.5 Separable First Order Equations

Given any real bivariate function $f(x, y)$ and any ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y), \tag{1.5.1}$$

we may certainly rewrite Equation (1.5.1) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$; however, this substitution is most fruitful in the case that $M(x, y) = M(x)$ is a function of the variable x alone and $N(x, y) = N(y)$ is a function of the variable y alone. Explicitly, in this case, Equation (1.5.1) can be viewed at last as

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{1.5.2}$$

We refer to such an ordinary differential equation as **separable**. Clarifying this terminology a bit, let us assume that y is a function of x. By the Chain Rule for Derivatives, it follows that

$$N(y)\frac{dy}{dx} = \frac{d}{dx}\int N(y)\,dy \text{ and } M(x) = \frac{d}{dx}\int M(x)\,dx.$$

Even though this analysis may seem either obscure or pointless, in fact, it provides crucial insight into solving separable ordinary differential equations since Equation (1.5.2) yields that

$$N(y)\,dy = -M(x)\,dx \text{ so that } \int N(y)\,dy = -\int M(x)\,dx.$$

By solving the left-hand side of the above integral equation, we obtain a solution to Equation (1.5.2).

Example 1.5.1. Prove that the following differential equation is separable; then, find a solution.

$$\frac{dy}{dx} = ye^x \text{ for all real numbers } y > 0$$

Solution. By dividing both sides of the above equation by y and subsequently subtracting e^x from both sides of the resulting equation, we find that that differential equation is separable of the form

$$-e^x + \frac{1}{y}\frac{dy}{dx} = 0.$$

By the above exposition, the differential equation can be solved by integrating as follows.

$$\ln(y) = \int \frac{1}{y}\,dy = \int e^x\,dx = e^x + C$$

Consequently, we find that $y(x) = e^{e^x + C} = Ce^{e^x}$ for some real number C. ◇

Example 1.5.2. Prove that the following differential equation is separable; then, find a solution.

$$\frac{dy}{dx} = y^2\cos^2(x)$$

Solution. Like before, it suffices to divide both sides of the above equation by y^2 and subsequently subtract $\cos^2(x)$ from the resulting equation. Carrying this out yields a separable equation

$$-\cos^2(x) + \frac{1}{y^2}\frac{dy}{dx} = 0.$$

We may now solve the differential equation by separating and integrating each side.

$$-\frac{1}{y} = \int \frac{1}{y^2}\,dy = \int \cos^2(x)\,dx = \frac{1}{2}\int [1 + \cos(2x)]\,dx = \frac{2x + \sin(2x) + C}{4}$$

By taking the reciprocal and flipping the sign, for some real number C, we find that

$$y(x) = -\frac{4}{2x + \sin(2x) + C}. \qquad \diamond$$

Example 1.5.3. Prove that the following differential equation is separable; then, find a solution.

$$\frac{x}{\ln(x)}\frac{dy}{dx} = 1 + y^2 \text{ for all real numbers } x > 0$$

Solution. Even though it does not appear on first glance that this equation is separable, by rearranging the quotient on the left-hand side and the sum on the right-hand side, we find that

$$\frac{1}{1+y^2}\frac{dy}{dx} = \frac{\ln(x)}{x}.$$

Consequently, the first order differential equation at hand is separable. By integrating, we find that

$$\arctan(y) = \int \frac{1}{1+y^2}\,dy = \int \frac{\ln(x)}{x}\,dx = \frac{[\ln(x)]^2}{2} + C.$$

Explicitly, one may use the substitution $u = \ln(x)$ with $du = \frac{1}{x}\,dx$ to solve in the antiderivative on the right-hand side. Ultimately, we conclude that the general solution to this equation is

$$y = \tan\left(\frac{[\ln(x)]^2}{2} + C\right). \qquad \diamond$$

Continuing with our hypothesis that y is a function of x, a different but equivalent analysis of the ordinary differential equation in Equation (1.5.2) is possible. We assume to this end that there exist real functions $R(x)$ and $S(y)$ such that $M(x) = R'(x)$ and $N(y) = S'(y)$, where the apostrophe denotes the derivative with respect to the attendant variable. By the Chain Rule, we have that

$$N(y)\frac{dy}{dx} = S'(y)\frac{dy}{dx} = \left[\frac{d}{dy}S(y)\right]\frac{dy}{dx} = \frac{d}{dx}[S(y)],$$

hence we may ultimately view Equation (1.5.2) as the derivative of the following with respect to x.

$$\frac{d}{dx}[R(x) + S(y)] = \frac{d}{dx}R(x) + \frac{d}{dx}S(y) = M(x) + N(y)\frac{dy}{dx} = 0 \tag{1.5.3}$$

Consequently, if we take the antiderivative of both sides of Equation (1.5.3) with respect to x, we obtain a general solution $R(x) + S(y) = C$ of the ordinary differential equation in Equation (1.5.2).

Example 1.5.4. Completely solve the initial value problem, providing the interval of definition.

$$1 - y^2 \frac{dy}{dx} = 0 \text{ and } y(0) = \sqrt[3]{3}$$

Solution. Observe that $x - \frac{1}{3}y^3 = C$ satisfies the given differential equation. By plugging in $x = 0$ and using the fact that $y(0) = \sqrt[3]{3}$, we may solve for C as follows.

$$C = 0 - \frac{1}{3}\left(\sqrt[3]{3}\right)^3 = -1$$

Consequently, the general solution of the above ordinary differential equation is given by

$$y(x) = \sqrt[3]{3x + 3}.$$

Considering that the domain of $y(x)$ is all real numbers, the interval of definition is $(-\infty, \infty)$. ◇

Example 1.5.5. Completely solve the initial value problem, providing the interval of definition.

$$\frac{dy}{dx} = \sqrt{y}\cos(x) \text{ and } y(0) = 0$$

Solution. Crucially, we note that the above differential equation is separable: it may be written as

$$-\cos(x) + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0.$$

Consequently, it follows that $-\sin(x) + 2\sqrt{y} = C$ satisfies the differential equation. Considering that $y(0) = 0$ and $\sin(0) = 0$, it follows that $C = 0$ so that $2\sqrt{y} = \sin(x)$; solving for y yields that

$$y(x) = \frac{\sin^2(x)}{4}.$$

Considering that the domain of $y(x)$ is all real numbers and the range of $\sin^2(x)$ is $[0, 1]$, it follows that $y(x) \geq 0$ for all real numbers x, hence we conclude that the interval of definition is $(-\infty, \infty)$. ◇

Example 1.5.6. Completely solve the initial value problem, providing the interval of definition.

$$\frac{1}{x}\frac{dy}{dx} = e^x \text{ and } y(0) = 1$$

Solution. Like all examples in this section, the above is a separable differential equation of the form

$$-xe^x + \frac{dy}{dx} = 0.$$

Consequently, we have that $-xe^x + e^x + y = C$ satisfies the differential equation. By plugging in $x = 0$ and using the fact that $y(0) = 1$ and $e^0 = 1$, it follows that $C = e^0 + 1 = 2$. We conclude that

$$y(x) = xe^x - e^x + 2.$$

Considering that the domain of $y(x)$ is all real numbers, the interval of definition is $(-\infty, \infty)$. ◇

1.6 Exact First Order Equations

Consider the following ordinary differential equation for some real functions $M(x, y)$ and $N(x, y)$.

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{1.6.1}$$

Back in Section 1.5, we established that if $M(x, y) = M(x)$ is a function of the variable x alone and $N(x, y) = N(y)$ is a function of the variable y alone, then the general solution of Equation (1.6.1) is particularly simple. Conversely, if there exists a real bivariate function $\psi(x, y)$ such that

$$\psi_x(x, y) = \frac{\partial}{\partial x}\psi(x, y) = M(x, y) \text{ and}$$
$$\psi_y(x, y) = \frac{\partial}{\partial y}\psi(x, y) = N(x, y),$$

then we say that Equation (1.6.1) is **exact**, and we refer to the above real bivariate function $\psi(x, y)$ as a **potential function** of the ordinary differential equation (1.6.1). Crucially, we require that the level curves $\psi(x, y) = k$ an explicit expression of $y = \varphi(x)$ as a differentiable function of x. Under this identification, by the Chain Rule for Partial Derivatives, it follows that

$$\frac{d}{dx}[\psi(x, \varphi(x))] = \psi_x(x, y) + \psi_y(x, y)\frac{dy}{dx} = M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

We omit the proof of the theorem below; the reader may find the details in [BD09, Theorem 2.6.1].

Theorem 1.6.1 (Criterion for the Existence of a Potential Function)**.** *Consider any pair of real bivariate functions $M(x, y)$ and $N(x, y)$ such that the partial derivatives $M_y(x, y)$ and $N_x(x, y)$ exist and are continuous on some open rectangle $(a, b) \times (c, d)$ in the Cartesian plane. We have that*

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \textit{ is exact if and only if } M_y(x, y) = N_x(x, y)$$

for every ordered pair (x, y) such that $a < x < b$ and $c < y < d$. Explicitly, provided that the partial derivatives $M_y(x, y)$ and $N_x(x, y)$ are equal on the open rectangle $(a, b) \times (c, d)$, then there exists a potential function $\psi(x, y)$ such that $\psi_x(x, y) = M(x, y)$ and $\psi_y(x, y) = N(x, y)$.

Even though we omit the proof of the above theorem, we provide a general approach for determining the potential function $\psi(x, y)$ underlying an exact differential equation as outlined in the proof.

Algorithm 1.6.2 (Constructing a Potential Function)**.** Consider any exact differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Carry out the following steps to determine the underlying potential function $\psi(x, y)$.

1.) By definition of a potential function, we have that $\psi_x(x, y) = M(x, y)$, hence it follows that

$$\psi(x, y) = f(y) + \int_{x_0}^{x} M(t, y)\, dt$$

for some real univariate function $f(y)$ that depends on y alone and some carefully chosen real number $a < x_0 < b$. We note that this holds by the Fundamental Theorem of Calculus.

2.) By definition of a potential function, we have that $\psi_y(x, y) = N(x, y)$, hence it follows that

$$N(x, y) = \psi_y(x, y) = f'(y) + \frac{\partial}{\partial y} \int_{x_0}^{x} M(t, y)\, dy$$

by the previous step, where $f'(y)$ denotes the derivative of f with respect to y.

3.) Use the previous step of the algorithm to solve for $f'(y)$; then, solve for $f(y)$ by integrating.

Caveat: to carry out the above algorithm for Constructing a Potential Function, it is necessary to begin with an exact equation; however, it is convenient and important to note that this algorithm can be fruitful for any ordinary differential equation of the form in Equation (1.6.1). Explicitly, if the function $f'(y)$ in the third step of the algorithm depends on x, then the equation is not exact.

Example 1.6.3. Prove that the following differential equation is exact; then, find a solution.

$$2x + 2y + (2x + 2y)\frac{dy}{dx} = 0$$

Solution. By the Criterion for the Existence of a Potential Function, it suffices to check that

$$\frac{\partial}{\partial y}(2x + 2y) = 2 = \frac{\partial}{\partial x}(2x + 2y),$$

hence the above differential equation is exact. Explicitly, we may find a potential function by first integrating $2x + 2y$ with respect to x and treating the constant of integration as a function of y.

$$\psi(x, y) = \int (2x + 2y)\, dx = x^2 + 2xy + f(y)$$

Even more, if we differentiate $\psi(x, y)$ with respect to y, we must obtain $2x + 2y$.

$$2x + 2y = \frac{\partial}{\partial y}\psi(x, y) = \frac{\partial}{\partial y}[x^2 + 2xy + f(y)] = 2x + f'(y).$$

Cancelling $2x$ from both sides yields that $f'(y) = 2y$ so that $f(y) = y^2$ and $\psi(x, y) = x^2 + 2xy + y^2$ is a potential function for this differential equation. Consequently, the general solution to the given differential equation is provided by $x^2 + 2xy + y^2 = \psi(x, y) = C$ for an arbitrary real number C. ⋄

Example 1.6.4. Prove that the following differential equation is exact; then, find a solution.

$$e^x y + \sin(y) + [e^x + x\cos(y)]\frac{dy}{dx} = 0$$

Solution. By the Criterion for the Existence of a Potential Function, it suffices to check that

$$\frac{\partial}{\partial y}[e^x y + \sin(y)] = e^x + \cos(y) = \frac{\partial}{\partial x}[e^x + x\cos(y)],$$

hence the differential equation is exact. Like before, the potential function is obtained by integrating $e^x y + \sin(y)$ with respect to x and treating the constant of integration as a function of y.

$$\psi(x, y) = \int [e^x y + \sin(y)]\, dx = e^x y + x\sin(y) + f(y).$$

Considering that the partial derivative of $\psi(x, y)$ with respect to y is $e^x + x\cos(y)$, we find that

$$e^x + x\cos(y) = \frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}[e^x y + x\sin(y) + f(y)] = e^x + x\cos(y) + f'(y).$$

Cancelling $e^x + x\cos(y)$ from both sides gives that $f'(y) = 0$ and $f(y)$ is constant — say $f(y) = 0$. We obtain the potential function $\psi(x, y) = e^x y + \sin(y)$, hence the general solution to the above differential equation is given by $e^x y + x\sin(y) = \psi(x, y) = C$ for an arbitrary real number C. ⋄

Example 1.6.5. Prove that the following differential equation is exact; then, find a solution.

$$\frac{y}{x} + \ln(y) + e^x + \left[\frac{x}{y} + \ln(x) + \cos(y)\right]\frac{dy}{dx} = 0$$

Solution. By taking the partial derivatives with respect to y and x, we find that

$$\frac{\partial}{\partial y}\left[\frac{y}{x} + \ln(y) + e^x\right] = \frac{1}{x} + \frac{1}{y} = \frac{\partial}{\partial x}\left[\frac{x}{y} + \ln(x) + \cos(y)\right],$$

hence the differential equation is exact. By integrating $\frac{y}{x} + \ln(y) + e^x$ in x, we find that

$$\psi(x,y) = \int \left[\frac{y}{x} + \ln(y) + e^x\right] dx = y\ln(x) + x\ln(y) + e^x + f(y).$$

By differentiating this equation with respect to y, we must have that

$$\frac{x}{y} + \ln(x) + \cos(y) = \frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}[y\ln(x) + x\ln(y) + e^x + f(y)] = \ln(x) + \frac{x}{y} + f'(y).$$

Cancelling the common terms from both sides of the equation yields that $f'(y) = \cos(y)$, and taking $f(y) = \sin(y)$ yields the potential function $\psi(x, y) = y\ln(x) + x\ln(y) + e^x + \sin(y)$. Consequently, the general solution is given by $y\ln(x) + x\ln(y) + e^x + \sin(y) = C$ for some real number C. ⋄

Example 1.6.6. Prove that the following differential equation is not exact.

$$e^x + e^y + (e^x + e^y)\frac{dy}{dx} = 0$$

Solution. By the Criterion for the Existence of a Potential Function, the given differential equation is not exact because the partial derivatives do not agree unless $y = x$. Explicitly, we have that

$$\frac{\partial}{\partial y}(e^x + e^y) = e^y \neq e^x = \frac{\partial}{\partial x}(e^x + e^y)$$

if $y \neq x$. On the other hand, if we assume that $y = x$, then the differential equation simplifies to

$$2e^x\left(1 + \frac{dy}{dx}\right) = 2e^x + 2e^x\frac{dy}{dx} = 0.$$

Considering that $2e^x$ is nonzero for all real numbers x, it follows that $1 + y' = 0$ and $y' = -1$. By taking the antiderivative with respect to x, we conclude that $y(x) = -x + C$ for some real number C. Clearly, this is a contradiction, hence the above differential equation is not exact. ⋄

Example 1.6.7. Prove that the following differential equation is not exact.

$$y\cos(xy) + x\sin(xy)\frac{dy}{dx} = 0$$

Solution. Compare the partial derivatives of the terms of the differential equation. We have that

$$\frac{\partial}{\partial y} y\cos(xy) = -xy\sin(xy) + \cos(xy) \text{ by the Product Rule and the Chain Rule and}$$
$$\frac{\partial}{\partial x} x\sin(xy) = xy\cos(xy) + \sin(xy) \text{ by the Product Rule and the Chain Rule.}$$

Consequently, the differential equation is exact if and only if the equation

$$-xy\sin(xy) + \cos(xy) = xy\cos(xy) + \sin(xy)$$

holds for all pairs of real numbers x and y. But this is demonstrably not the case: by taking $x = 0$ and $y = 1$, the left-hand side of the above equation is 1, but the right-hand side is 0. ◇

Even if our ordinary differential equation does not appear exact on first glance, it is possible in some cases to introduce an integrating factor that resolves in an exact equation. Explicitly, suppose that there exists a real bivariate function $\mu(x, y)$ such that the following equation is exact.

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0$$

By the Criterion for the Existence of a Potential Function, it is necessary that

$$\frac{\partial}{\partial y}\mu(x,y)M(x,y) = \frac{\partial}{\partial x}\mu(x,y)N(x,y). \tag{1.6.2}$$

By the Product Rule for Derivatives, the above displayed equation resolves to

$$\mu_y(x,y)M(x,y) + \mu(x,y)M_y(x,y) = \mu_x(x,y)N(x,y) + \mu(x,y)N_x(x,y).$$

Unfortunately, in general, Equation (1.6.2) is a partial differential equation; however, if we assume that $\mu(x, y) = \mu(x)$ is a function of the variable x alone, then it follows that $\mu_y(x, y) = 0$ so that

$$\frac{d}{dx}\mu(x) = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)}\mu(x). \tag{1.6.3}$$

Crucially, if the coefficient of $\mu(x)$ in the above Equation (1.6.3) is a function of x alone, then this displayed equation can be viewed as a linear and separable first order ordinary differential equation, hence it can be solved with methods we have previously discussed. Likewise, one can perform a similar analysis in the case that $\mu(x, y) = \mu(y)$ is a function of the variable y alone to find that

$$\frac{d}{dy}\mu(y) = \frac{N_x(x,y) - M_y(x,y)}{M(x,y)}\mu(y).$$

Algorithm 1.6.8 (Constructing an Exact Integrating Factor)**.** Consider any differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

Carry out the following steps to determine an integrating factor $\mu(x, y)$ that resolves in exactness.

1.) Compute the partial derivatives $M_y(x, y)$ and $N_x(x, y)$.

2.) Compute the following pair of quotients of functions.

$$P(x,y) = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)} \text{ and } Q(x,y) = \frac{N_x(x,y) - M_y(x,y)}{M(x,y)}$$

3.) If $P(x, y) = P(x)$ is a function of the variable x alone, then the ordinary differential equation

$$\frac{d}{dx}\mu(x) = P(x)\mu(x)$$

is linear and separable; solve this differential equation to obtain the integrating factor $\mu(x)$.

4.) If $Q(x, y) = Q(y)$ is a function of the variable y alone, then the ordinary differential equation

$$\frac{d}{dy}\mu(y) = Q(y)\mu(y)$$

is linear and separable; solve this differential equation to obtain the integrating factor $\mu(y)$.

Example 1.6.9. Prove that the following differential equation is not exact; construct an integrating factor that resolves the equation to an exact equation; and ultimately, solve the equation.

$$e^x + e^y + (e^x - e^y)\frac{dy}{dx} = 0$$

Solution. Comparing the partial derivatives of the terms of the differential equation, we find that

$$e^y = \frac{\partial}{\partial y}(e^x + e^y) = \frac{\partial}{\partial x}(e^x - e^y) = e^x$$

if and only if $y = x$. But as in Example 1.6.6, this is impossible: indeed, if we assume that $y = x$, then the given differential equation simplifies to $2e^x = 0$ — a contradiction. Consequently, we are not dealing with an exact equation; however, as $M_y(x, y) = e^y$ and $N_x(x, y) = e^x$, it follows that

$$P(x,y) = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{e^y - e^x}{e^x - e^y} = -1$$

is a function of the variable x alone. By the algorithm for Constructing an Exact Integrating Factor, in order to determine an integrating factor $\mu(x)$, it suffices to solve the differential equation

$$\frac{d}{dx}\mu(x) = -\mu(x).$$

By rewriting this equation as $\frac{1}{\mu}d\mu = -dx$, it follows by basic antidifferentiation that $\ln(\mu) = -x$ so that $\mu(x) = e^{-x}$. By multiplying the original equation by $\mu(x) = e^{-x}$, we obtain an exact equation

$$1 + e^{y-x} + (1 - e^{y-x})\frac{dy}{dx} = 0.$$

Explicitly, we have that $\frac{\partial}{\partial y}(1 + e^{y-x}) = e^{y-x}$ and $\frac{\partial}{\partial x}(1 - e^{y-x}) = e^{y-x}$ by the Chain Rule. Consequently, according to the algorithm for Constructing a Potential Function, it follows that

$$\psi(x,y) = \int (1 + e^{y-x})\,dx = x - e^{y-x} + f(y).$$

By taking the derivative of $\psi(x,y)$ with respect to y, we find that

$$1 - e^{y-x} = \frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}[x - e^{y-x} + f(y)] = -e^{y-x} + f'(y).$$

Cancelling term $-e^{y-x}$ from both sides of this equation yields that $f'(y) = 1$ so that $f(y) = y$. We obtain the general solution $x + y - e^{y-x} = \psi(x,y) = C$ for an arbitrary real number C. ◇

Example 1.6.10. Prove that the following differential equation is not exact; construct an integrating factor that resolves the equation to an exact equation; and ultimately, solve the equation.

$$2xy^3 + y^4 + (xy^3 - 2y)\frac{dy}{dx} = 0$$

Solution. Comparing the partial derivatives of the terms of the differential equations, we find that

$$\frac{\partial}{\partial y}(2xy^3 + y^4) = 6xy^2 + 4y^3 \text{ and } \frac{\partial}{\partial x}(xy^3 - 2y) = y^3.$$

Consequently, the partial derivatives are not equal as functions, hence the differential equation is not exact; however, our computations yield that $M_y(x,y) = 6xy^2 + 4y^3$ and $N_x(x,y) = y^3$ so that

$$Q(x,y) = \frac{N_x(x,y) - M_y(x,y)}{M(x,y)} = \frac{y^3 - (6xy^2 + 4y^3)}{2xy^3 + y^4} = \frac{-3y^2(2x+y)}{y^3(2x+y)} = -\frac{3}{y}.$$

By the algorithm for Constructing an Exact Integrating Factor, we may find an integrating factor $\mu(y)$ by solving the following first order linear differential equation.

$$\frac{d}{dy}\mu(y) = -\frac{3}{y}\mu(y).$$

By separating the terms and performing basic antidifferentiation, we find that

$$\ln[\mu(y)] = \int \frac{1}{\mu(y)}\,d\mu(y) = -3\int \frac{1}{y}\,dy = -3\ln(y)$$

so that $\mu(y) = y^{-3}$, and multiplying the original equation by $\mu(y) = y^{-3}$ yields an exact equation

$$2x + y + \left(x - \frac{2}{y^2}\right)\frac{dy}{dx} = 0.$$

Continuing with the algorithm for Constructing a Potential Function, we have that

$$\psi(x,y) = \int (2x + y)\,dx = x^2 + xy + f(y).$$

By taking the derivative with respect to y of $\psi(x,y)$, it follows that

$$x - \frac{2}{y^2} = \frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}[x^2 + xy + f(y)] = x + f'(y).$$

Consequently, we obtain that $f'(y) = -2y^{-2}$ so that $f(y) = 2y^{-1}$. We conclude that the general solution is given implicitly by $x^2 + xy + 2y^{-1} = \psi(x,y) = C$ for an arbitrary real number C. ◇

Example 1.6.11. Prove that the following differential equation is not exact; then, explain why the algorithm for Constructing an Exact Integrating Factor fails to produce a solution in this case.

$$x^2 - xy + (xy + y^2)\frac{dy}{dx} = 0$$

Solution. Comparing the partial derivatives of the terms of the differential equations, we find that

$$\frac{\partial}{\partial y}(x^2 - xy) = -x \text{ and } \frac{\partial}{\partial x}(xy + y^2) = y.$$

Consequently, the partial derivatives are not equal as functions, hence the differential equation is not exact; however, our computations yield that $M_y(x, y) = -x$ and $N_x(x, y) = y$ so that

$$P(x, y) = \frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \frac{-(x + y)}{xy + y^2} = -\frac{1}{y} \text{ and}$$

$$Q(x, y) = \frac{N_x(x, y) - M_y(x, y)}{M(x, y)} = \frac{x + y}{x^2 - xy}.$$

Unfortunately, neither $P(x, y)$ is a function of x nor $Q(x, y)$ is a function of y, hence the algorithm for constructing an exact integrating factor fails. Even worse, it turns out that the given differential equation does not admit a solution that can be expressed in terms of elementary functions! ⋄

1.7 Existence and Uniqueness of Solutions

Back in Remark 1.3.13, it was pointed out that there exist differential equations that cannot be solved. We encountered such an instance in Example 1.6.11. Even those differential equations that admit solutions might have an abundance of them, and worse yet, it is typically impossible to realize solutions of differential equations in terms of elementary functions. We turn our attention next to the criteria that determine the existence and uniqueness of solutions to differential equations. We leave the details of the following theorem to the curious reader (see [BD09, Theorem 2.4.1]).

Theorem 1.7.1 (Fundamental Theorem of First Order Linear Ordinary Differential Equations)**.** *Consider any real functions $p(t)$ and $g(t)$ that are continuous on an open interval (a, b) containing a point t_0. Given any real number y_0, there exists one and only one function $y = \phi(t)$ satisfying that*

$$\frac{dy}{dt} + p(t)y = g(t)$$

on (a, b) and $y(t_0) = y_0$. Put another way, if $p(t)$ and $g(t)$ are continuous on an open interval, then the above first order linear ordinary differential equation admits a unique solution $y = \phi(t)$.

Even more, the proof of the Fundamental Theorem of First Order Linear Ordinary Differential Equations provides a closed form expression for the solution to the above differential equation.

Corollary 1.7.2 (Leibniz Formula for Solutions of First Order Linear Ordinary Differential Equations)**.** *Consider any real univariate functions $p(t)$ and $g(t)$ that are continuous on an open interval (a, b) containing t_0. Given any real number y_0, consider the functions $\mu(t) = \exp\Big(\int_{t_0}^t p(s)\,ds\Big)$ and*

$$\phi(t) = \frac{1}{\mu(t)}\bigg(y_0 + \int_{t_0}^t \mu(s)g(s)\,ds\bigg).$$

We have that $\phi(t)$ is the unique solution of the initial value problem on (a, b) with $y(t_0) = y_0$ and

$$\frac{dy}{dt} + p(t)y = g(t).$$

Example 1.7.3. Construct an open interval on which the following differential equation admits a unique solution; then, determine the closed form expression for the solution with $y(1) = 2$.

$$\frac{dy}{dt} + \frac{1}{t-3}y = 2t$$

Solution. By the Fundamental Theorem of First Order Linear Ordinary Differential Equations, it suffices to determine an open interval on which both of the functions $p(t) = 1/(t-3)$ and $g(t) = 2t$ are continuous. Considering that $g(t)$ is a linear polynomial and hence everywhere continuous, we conclude that the largest interval on which $p(t)$ and $g(t)$ are both continuous is $(0, 3) \cup (3, \infty)$. We may restrict our attention to the open interval $(0, 3)$ in light of the fact that we wish to solve the initial value problem with $t_0 = 1$. Corollary 1.7.2 suggests we compute the functions

$$\mu(t) = \exp\bigg(\int_{t_0}^t p(s)\,ds\bigg) \text{ and } \phi(t) = \frac{1}{\mu(t)}\bigg(y_0 + \int_{t_0}^t \mu(s)g(s)\,ds\bigg)$$

since the latter is the unique solution to the given initial value problem. By doing so, we find that

$$\mu(t) = \exp\bigg(\int_1^t \frac{1}{s-3}\,ds\bigg) = \exp[\ln(3-s)]_1^t = \frac{1}{2}(3-t) \text{ so that}$$

$$\phi(t) = \frac{2}{3-t}\bigg(2 + \int_1^t s(3-s)\,ds\bigg) = \frac{2}{3-t}\bigg(2 + \int_1^t (3s - s^2)\,ds\bigg) = \frac{2}{3-t}\bigg(-\frac{1}{3}t^3 + \frac{3}{2}t^2 + \frac{5}{6}\bigg). \quad \diamond$$

Example 1.7.4. Construct an open interval on which the following differential equation admits a unique solution; then, determine the closed form expression for the solution with $y(\pi) = 1$.

$$\frac{dy}{dt} + \tan(t)y = \sin(t)$$

Solution. Observe that $\tan(t)$ is continuous for all real numbers other than those of the form $\frac{(2n+1)\pi}{2}$ for some integer n. Even more, $\sin(t)$ is everywhere continuous. Consequently, it follows by the Fundamental Theorem of First Order Linear Ordinary Differential Equations that the above differential equations admits a solution for all real numbers $t \neq \frac{(2n+1)\pi}{2}$ for any integer n. Considering that we are in search of a solution in the case that $t_0 = \pi$, we may consider the open interval $\big(\frac{\pi}{2}, \frac{3\pi}{2}\big)$. Corollary 1.7.2 guarantees that the unique solution to the given initial value problem depends on

$$\mu(t) = \exp\bigg(\int_{t_0}^t p(s)\,ds\bigg) = \exp\bigg(\int_\pi^t \tan(s)\,ds\bigg) = \exp[\ln|\sec(t)|] = -\sec(t).$$

We note that the resulting function is negative because $\sec(t)$ is negative on the open interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Carrying this computation forward through the corollary, we find that

$$\phi(t) = \frac{1}{\mu(t)}\left(y_0 + \int_{t_0}^{t} \mu(s)g(s)\,ds\right) = \cos(t)\left(1 + \int_{\pi}^{t} \tan(s)\,ds\right) = \cos(t) + \cos(t)\ln|\sec(t)|$$

is the unique solution to the above initial value problem. ◇

Even more, the following theorem completely resolves the question of existence and uniqueness of solutions to first order non-linear ordinary differential equations and generalizes 1.7.1.

Theorem 1.7.5 (Fundamental Theorem of First Order Ordinary Differential Equations)**.** *Consider any real bivariate function $f(t, y)$ such that both $f(t, y)$ and its partial derivative $\partial f/\partial y$ are continuous on an open rectangle $(a, b) \times (c, d)$ containing a point (t_0, y_0). We may find some real number h for which there exists an open interval $(t_0 - h, t_0 + h)$ contained within the open interval (a, b) such that there exists one and only one function $y = \phi(t)$ satisfying that $y(t_0) = y_0$ and*

$$\frac{dy}{dt} = f(t, y).$$

Put another way, if $f(t, y)$ and its partial derivative with respect to y are continuous on an open rectangle, then the above first order ordinary differential equation admits a unique solution $y = \phi(t)$.

Example 1.7.6. Construct an open rectangle on which the following differential equation admits a unique solution; then, determine the closed form expression for the solution with $y(1) = 2$.

$$\frac{dy}{dt} + y^3 = 0$$

Solution. Considering that the above differential equation can be rewritten as $y' = -y^3$, it follows that $f(t, y) = -y^3$ and $f_y = -3y^2$ are both continuous, hence there exists a real number h such that the above initial value problem admits a unique solution on $(1 - h, 1 + h) \times (-\infty, \infty)$ by the Fundamental Theorem of First Order Ordinary Differential Equations. We construct the unique solution (determining the value of the real number h) as follows: indeed, we have that

$$\frac{dy}{dt} = -y^3 \text{ if and only if } -\frac{dy}{y^3} = dt,$$

hence integrating both sides of this equation yields that $1/2y^2 = t + C$. By plugging in $t = 1$ and using the fact that $y(1) = 2$, it follows that $C + 1 = 1/8$ so that $C = -7/8$. Consequently, the solutions to the above initial value problem are given by $y^2 = 2/(t - 7/8)$. Even more, if we assume that y is positive, then the unique solution of the initial value problem is $\phi(t) = \sqrt{2/(t - 7/8)}$. Observe that $\phi(t)$ is defined for all real numbers $t > 7/8$, hence we may choose $h = 1/8$. ◇

Example 1.7.7. Compute the largest region in the ty-plane for which the hypotheses of the Fundamental Theorem of First Order Ordinary Differential Equations are satisfied for the following.

$$\frac{dy}{dt} = \sqrt{1 - t^2 - y^2}$$

Solution. Observe that the real bivariate function $f(t, y) = \sqrt{1 - t^2 - y^2}$ is defined if and only if $1 - t^2 - y^2 \geq 0$ if and only if $t^2 + y^2 \leq 1$. Even more, its first partial derivative with respect to y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(1 - t^2 - y^2)^{1/2} = \frac{-2y}{(1 - t^2 - y^2)^{1/2}}$$

is defined if and only if $1 - t^2 - y^2 > 0$ if and only if $t^2 + y^2 < 1$. Both functions are continuous on their domains because they are products and compositions of continuous functions. ◇

Example 1.7.8. Compute the largest region in the ty-plane for which the hypotheses of the Fundamental Theorem of First Order Ordinary Differential Equations are satisfied for the following.

$$\frac{dy}{dt} = \ln(y^2 - t)$$

Solution. Observe that the real bivariate function $f(t, y) = \ln(y^2 - t)$ is defined if and only if $y^2 - t > 0$ if and only if $y^2 > t$. Even more, its first partial derivative with respect to y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \ln(y^2 - t) = \frac{2y}{y^2 - t}$$

is defined if and only if $y^2 - t$ is nonzero. Both functions are continuous on their domains, hence the region in question consists of all ordered pairs of real numbers (t, y) such that $y^2 > t$. ◇

Unfortunately, it is entirely possible that the first order non-linear ordinary differential equation $y' = f(t, y)$ cannot be solved by any of the methods outlined in the previous sections of this chapter; however, in this case, it is sometimes possible to carry out the following algorithm successfully.

Algorithm 1.7.9 (Picard's Method)**.** Consider any first order ordinary differential equation

$$\frac{dy}{dt} = f(t, y).$$

Carry out the following steps to determine a solution $y = \phi(t)$ of the initial value problem $y(0) = 0$.

1.) Begin with the real function $\phi_0(t) = 0$.

2.) Construct the real function $\phi_1(t) = \int_0^t f[s, \phi_0(s)]\, ds$.

3.) Construct the real function $\phi_2(t) = \int_0^t f[s, \phi_1(s)]\, ds$.

4.) Continue to define the real function $\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds$ for each integer $n \geq 2$.

Provided that the sequence $\{\phi_n(t)\}_{n \geq 0}$ of real functions converges to a differentiable function $\phi(t)$ satisfying that $\phi'(t) = f(t, y)$ and $\phi(0) = 0$, we have obtained a solution to the differential equation.

Example 1.7.10. Employ Picard's Method to solve the initial value problem $y(0) = 0$ and

$$\frac{dy}{dt} = 2(y + 1).$$

Solution. Considering that $f(t, y) = 2(y + 1)$ does not depend at all on the variable t, the problem is quite simple: indeed, we have that $f[s, \phi_n(s)] = 2(\phi_n(s) + 1)$ for all integers $n \geq 0$ so that

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds = 2\int_0^t (\phi_n(s) + 1)\, ds = 2t + 2\int_0^t \phi_n(s)\, ds$$

for each integer $n \geq 0$. By Picard's Method, it follows that $\phi_0(t) = 0$ so that $\phi_1(t) = 2t$,

$$\phi_2(t) = 2t + 2\int_0^t \phi_1(s)\, ds = 2t + 4\int_0^t s\, ds = 2t + 2t^2,$$

$$\phi_3(t) = 2t + 2\int_0^t \phi_2(s)\, ds = 2t + 4\int_0^t (s + s^2)\, ds = 2t + 2t^2 + \frac{4}{3}t^3,$$

$$\phi_4(t) = 2t + 2\int_0^t \phi_3(s)\, ds = 2t + 4\int_0^t \left(s + s^2 + \frac{2}{3}s^3\right) ds = 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}s^4, \text{ and}$$

$$\phi_5(t) = 2t + 2\int_0^t \phi_4(s)\, ds = 2t + 4\int_0^t \left(s + s^2 + \frac{2}{3}s^3 + \frac{1}{3}s^4\right) ds = 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}s^4 + \frac{4}{3 \cdot 5}s^5.$$

Cleverly noting the structure of the denominator of the last summand of $\phi_5(t)$, we find that

$$\phi_5(t) = \frac{2^1}{1!}t + \frac{2^2}{2!}t^2 + \frac{2^3}{3!}t^3 + \frac{2^4}{4!}t^4 + \frac{2^5}{5!}s^5,$$

from which we surmise the general form of the nth iterate of Picard's Method as follows.

$$\phi_n(t) = \sum_{k=1}^{n} \frac{2^k}{k!}t^k.$$

(We could verify this formula using mathematical induction, but that is beyond the scope of this class.) Bearing this in mind, it follows that the limit of this sequence of functions is the power series

$$\phi(t) = \lim_{n \to \infty} \phi_n(t) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{2^k}{k!}t^k = \sum_{k=1}^{\infty} \frac{2^k}{k!}t^k = \sum_{k=1}^{\infty} \frac{(2t)^k}{k!} = e^{2t} - 1.$$

Clearly, this function is differentiable with $\phi'(t) = 2e^{2t} = 2(\phi(t) + 1)$ and $\phi(0) = 0$. ◇

Example 1.7.11. Employ Picard's Method to solve the initial value problem $y(0) = 0$ and

$$\frac{dy}{dt} = ty + 1.$$

Solution. Beginning with $f(t, y) = ty + 1$ and $\phi_0(t) = 0$, it follows by Picard's Method that

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds = \int_0^t (s\phi_n(s) + 1)\, ds = t + \int_0^t s\phi_n(s)\, ds$$

for each integer $n \geq 0$. Consequently, it suffices to determine the definite integral $\int_0^t s\phi_n(s)\,ds$ for each integer $n \geq 0$. Certainly, if $n = 0$, this integral evaluates to 0, hence we find that $\phi_1(t) = t$. We are now in a fine position to determine the general form for the nth iterate of Picard's Method.

$$\phi_2(t) = t + \int_0^t s\phi_1(s)\,ds = t + \int_0^t s^2\,ds = t + \frac{1}{3}t^3$$

$$\phi_3(t) = t + \int_0^t s\phi_2(s)\,ds = t + \int_0^t \left(s^2 + \frac{1}{3}s^4\right)ds = t + \frac{1}{3}t^3 + \frac{1}{3 \cdot 5}t^5$$

$$\phi_4(t) = t + \int_0^t s\phi_3(s)\,ds = t + \int_0^t \left(s^2 + \frac{1}{3}s^4 + \frac{1}{3 \cdot 5}s^6\right)ds = t + \frac{1}{3}t^3 + \frac{1}{3 \cdot 5}t^5 + \frac{1}{3 \cdot 5 \cdot 7}t^7$$

We surmise from the first four terms of the sequence that the general term satisfies that

$$\phi_n(t) = \sum_{k=1}^{n} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-1)} t^{2k-1}.$$

(Once again, mathematical induction could be employed to establish the above formula.) Considering that this sequence constitutes the partial sums of a power series, it converges to the function

$$\phi(t) = \lim_{n \to \infty} \phi_n(t) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-1)} t^{2k-1} = \sum_{k=1}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-1)} t^{2k-1}.$$

Even more, it follows that $\phi(t)$ is differentiable and $\phi(0) = 0$. Checking the derivative yields that

$$\begin{aligned}
\phi'(t) = \frac{d}{dt} \sum_{k=1}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-1)} t^{2k-1} &= \sum_{k=1}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-1)} \frac{d}{dt} t^{2k-1} \\
&= \sum_{k=1}^{\infty} \frac{2k-1}{1 \cdot 3 \cdot 5 \cdots (2k-1)} t^{2k-2} \\
&= \sum_{k=1}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-3)} t^{2k-2} \\
&= 1 + \sum_{k=2}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-3)} t^{2k-2} \\
&= 1 + t\sum_{k=2}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k-3)} t^{2k-3} = t\phi(t) + 1. \qquad \diamond
\end{aligned}$$

Each of the above examples could be computed by elementary means because they are both separable equations, hence we could determine a closed form for the power series in Example 1.7.11.

Chapter 2

Second Order Linear Equations

Even more fundamental to the study of mathematical physics than first order differential equations are the second order differential equations that model such physical phenomena as fluid mechanics, heat conduction, wave motion, or electromagnetism. Besides these applications, the theory of second order linear differential equations enjoys a beautiful and systematic approach that encompasses and brings together many critical areas of mathematics, e.g., the algebraic study of roots of polynomials, determinants and systems of equations from linear algebra, and power series from calculus.

2.1 Homogeneous Linear Equations: Constant Coefficients

By definition, a second order ordinary differential equation is any differential equation of the form

$$F(t, y, y', y'') = 0$$

for some real function F. Even more, we say that the above differential equation is linear provided that F is a linear function, i.e., there exist real functions $G(t)$, $P(t)$, $Q(t)$, and $R(t)$ such that

$$F(t, y, y', y'') = G(t) - P(t)y'' - Q(t)y' - R(t)y.$$

We note the significance of the assumption that the functions G, P, Q, and R depend only on the variable t, hence F is a linear function in y, y', and y''. Certainly, if the function $P(t)$ is zero for all real numbers t, then the above differential equation does not possess order two, hence we may assume that $P(t)$ is nonzero for some real number t. By the same rationale, we need only consider the real numbers t for which $P(t)$ is nonzero, hence we may rewrite the differential equation as

$$y'' + \frac{Q(t)}{P(t)}y' + \frac{R(t)}{P(t)}y = \frac{G(t)}{P(t)}.$$

We will therefore restrict our attention to the open intervals where the above functions of t are continuous. Like in the previous chapter, the initial value problems in second order ordinary differential equations amount to solving a differential equation subject to the constraints that $y(t_0) = y_0$ and $y'(t_0) = y_0'$ for some real numbers t_0, y_0, and y_0'. Crucially, we need two initial conditions that determine an ordered pair (t_0, y_0) and the slope y_0' of $y(t)$ at the point t_0. Essentially, this is due to the Fundamental Theorem of Calculus: to obtain $y(t)$ from $y''(t)$, we must antidifferentiate twice, resulting in up to two constants of integration for which up to two equations are required to solve. We say that a second order linear ordinary differential equation is **homogeneous** if

$$P(t)y'' - Q(t)y' + R(t)y = 0$$

for some real functions P, Q, and R; otherwise, the differential equation is **non-homogeneous**.

We will fix our attention throughout this section on the case that the functions P, Q, and R are constant and the second order linear ordinary differential equation is homogeneous. Explicitly, we will assume that there exist real numbers a, b, and c such that a is nonzero and

$$ay'' + by' + cy = 0. \tag{2.1.1}$$

Roughly speaking, any solution to the above equation must be a function $y(t)$ that satisfies a linear relation with its first and second derivatives. We illustrate this notion as follows: if $b = 0$, then $ay'' + cy = 0$ if and only if $y'' = -\frac{c}{a}y$. Consequently, the second derivative of y must be a constant multiple of y. Considering the gamut of elementary functions from calculus and their derivatives, the astute reader may notice that an exponential function of the form e^{rt} for some real number t has second derivative r^2e^{rt}, hence the second derivative of e^{rt} is in fact a constant multiple of e^{rt}.

Bearing this in mind, we use the exponential function e^{rt} as a prototype for Equation (2.1.1). By substituting $y = e^{rt}$, $y' = re^{rt}$, and $y'' = r^2e^{rt}$, we obtain a homogeneous equation

$$(ar^2 + br + c)e^{rt} = 0.$$

Considering that the exponential function is never zero, it follows that

$$ar^2 + br + c = 0. \tag{2.1.2}$$

Equation (2.1.2) is called the **characteristic equation** of Equation (2.1.1). Crucially, the roots of the characteristic equation correspond to solutions of a second order homogeneous linear ordinary differential equation with constant coefficients, hence we have reduced a seemingly difficult question of calculus to simply checking the roots of a quadratic polynomial. We are afforded to this end the powerful tool of the Quadratic Formula; however, we must consider the following three cases.

1.) Equation (2.1.2) admits two distinct real solutions.

2.) Equation (2.1.2) admits two identical real solutions.

3.) Equation (2.1.2) admits two non-real complex solutions.

Beginning with the first case, we will deal with each of the above scenarios in turn. We will assume to this end that r_1 and r_2 are two distinct real solutions of the characteristic equation of a second order homogeneous linear ordinary differential equation. Consequently, if we assume that $y_1 = e^{r_1t}$, then $y_1' = r_1e^{r_1t}$ and $y_1'' = r_1^2e^{r_1t}$ together with the fact that $ar_1^2 + br_1 + c = 0$ imply that

$$ay_1'' + by_1' + cy_1 = (ar_1^2 + br_1 + c)e^{r_1t} = 0.$$

Likewise, the same analysis holds for $y_2 = e^{r_2t}$, hence we conclude that for any real numbers c_1 and c_2, the **linear combination** $y(t) = c_1y_1(t) + c_2y_2(t)$ of the functions $y_1(t)$ and $y_2(t)$ satisfies that

$$ay'' + by' + cy = 0$$

by the linearity of the ordinary derivative. Even more, if we impose some initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$, then we obtain the following system of equations.

$$\begin{cases} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0' \end{cases}$$

By solving for the real number c_2 in the first equation and substituting it into the second equation, we may solve for the real number c_1 in terms of r_1, r_2, t_0, y_0, and y_0'; then, this expression can be substituted into the equation for c_2 to obtain an expression in terms of r_1, r_2, t_0, y_0, and y_0'. We omit the details of this process; however, the upshot is that the initial value problem is solved by

$$\begin{cases} c_1 = \dfrac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0} \text{ and} \\ c_2 = \dfrac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}. \end{cases}$$

Often, in practice, the process of solving the above system of equations will be much clearer.

Algorithm 2.1.1 (Solutions of Homogeneous Second Order Linear Ordinary Differential Equation with Constant Coefficients I)**.** Given any real univariate function $y = f(t)$ and any real numbers a, b, and c such that a is nonzero, consider the homogeneous second order linear differential equation

$$ay'' + by' + c = 0.$$

Carry out the following steps to determine the general solution $y = \phi(t)$.

1.) Compute the roots of the characteristic equation $ar^2 + br + c = 0$.

2.) Provided that the above characteristic equation admits two distinct real solutions r_1 and r_2, the general solution of the above differential equation is given by $\phi(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

3.) Otherwise, if the characteristic equation admits two identical real solutions or two complex conjugate solutions, then further analysis of the differential equation is required.

Given any initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$,

4.) solve the following system of equations to obtain the solution of the initial value problem.

$$\begin{cases} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0' \end{cases}$$

Example 2.1.2. Construct the general solution to the following differential equation.

$$y'' - 2y' - 3y = 0$$

Solution. Considering that the characteristic polynomial $r^2 - 2r - 3$ factors into distinct linear polynomials $(r+1)(r-3)$, it follows that $r_1 = -1$ and $r_2 = 3$ are its distinct real roots. Consequently, the general solution of the above differential equation is given by $y(t) = c_1 e^{-t} + c_2 e^{3t}$. ◇

Example 2.1.3. Construct the solution to the initial value problem with $y(0) = 1$, $y'(0) = 8$, and

$$y'' - 7y' + 10y = 0.$$

Solution. Like in the above example, the characteristic polynomial $r^2 - 7r + 10$ factors into distinct linear polynomials $(r - 5)(r - 2)$, hence the general solution of the above differential equation is $y(t) = c_1 e^{5t} + c_2 e^{2t}$. Considering that $y(0) = 1$ and $y'(0) = 8$, we obtain a system of equations.

$$\begin{cases} c_1 + c_2 = 1 \\ 5c_1 + 2c_2 = 8 \end{cases}$$

By subtracting twice the first equation from the second equation, we find that $3c_1 = 6$ so that $c_1 = 2$ and $c_2 = 1 - c_1 = -1$. We conclude that $y(t) = 2e^{5t} - e^{2t}$ is the desired solution. ◇

Example 2.1.4. Construct the solution to the initial value problem with $y(0) = 2$, $y'(0) = -2$, and

$$6y'' - 13y' + 6y = 0.$$

Solution. Observe that $6r^2 - 13r + 6 = (2r - 3)(3r - 2)$, hence the underlying characteristic equation $6r^2 - 13r + 6 = 0$ admits two distinct real solutions $r_1 = 3/2$ and $r_2 = 2/3$. We obtain the general solution of the above differential equation as $y(t) = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{2}{3}t}$; in order to obtain the particular solution of the given initial value problem, we must solve the system of equations below.

$$\begin{cases} c_1 + c_2 = 2 \\ \dfrac{3}{2}c_1 + \dfrac{2}{3}c_2 = -2 \end{cases}$$

Clearing the coefficient of $2/3$ from c_2 in the second equation yields that $9c_2/4 + c_2 = -3$; then, subtracting the first equation from this equation, we find that $5c_1/4 = -5$. We conclude that $c_1 = -4$ so that $c_2 = 2 - c_1 = 6$, hence the solution we seek is $y(t) = -4e^{\frac{3}{2}t} + 6e^{\frac{2}{3}t}$. ◇

2.2 Homogeneous Linear Equations and the Wronskian

Continuing our discussion of second order linear ordinary differential equations, let us assume that $y = f(t)$ is a real univariate function and $g(t)$, $p(t)$, and $q(t)$ are continuous real functions defined on an open interval I that contains a real number t_0. Consider the initial value problem

$$y'' + p(t)y' + q(t) = g(t) \tag{2.2.1}$$

with $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Last section, in Algorithm 2.1.1 and the discussion preceding it, we outlined a method for solving this initial value problem in the case that $p(t)$ and $q(t)$ are constant functions and $g(t)$ is zero; however, the observant reader may very well find the exposition on the matter unsatisfactory: indeed, it is entirely natural to wonder why all possible solutions of such a differential equation are accounted for by the aforementioned algorithm. We turn our attention in this section to this question; it is completely answered by the following fundamental theorem.

Theorem 2.2.1 (Fundamental Theorem of Second Order Linear Ordinary Differential Equations). *Consider the initial value problem with* $y(t_0) = y_0$, $y'(t_0) = y_0'$, *and*

$$y'' + p(t)y' + q(t)y = g(t)$$

for some real functions $g(t)$, $p(t)$, *and* $q(t)$ *that are continuous on an open interval* I *that contains the point* t_0. *There exists exactly one real univariate function* $\phi(t)$ *that is defined for all real numbers* t *in the open interval* I *and satisfies the initial value problem, i.e.,* $\phi(t_0) = y_0$, $\phi'(t_0) = y_0'$, *and*

$$\phi'' + p(t)\phi' + q(t)\phi = g(t).$$

Bluntly put, the proof of the Fundamental Theorem of Second Order Linear Ordinary Differential Equations is far beyond the scope of this course; however, we will simply accept it as a black box.

Example 2.2.2. Construct an open interval of maximum length for which the initial value problem with $y(\pi) = 1$ and $y'(\pi) = 0$ admits a solution, but do not attempt to solve the equation.

$$\cos(t)y'' + \sin(t)y' + \cos(t)y = 0$$

Solution. Considering that $\cos(\pi) = 1$, we may divide by $\cos(t)$ to obtain an equation of the form

$$y'' + \tan(t)y' + y = 0.$$

Considering that $\tan(t)$ is continuous on its domain, it follows that $p(t) = \tan(t)$, $q(t) = 1$, and $g(t) = 0$ are continuous on the open interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. By the Fundamental Theorem of Second Order Linear Ordinary Differential Equations, this is the open interval of maximum length for which the initial value problem admits a solution since $\tan(t)$ is not defined for $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$. ◇

Example 2.2.3. Construct an open interval of maximum length for which the initial value problem with $y(1) = e$ and $y'(1) = 3.14$ admits a solution, but do not attempt to solve the equation.

$$t^2y'' + ty' + y = \ln|t|$$

Solution. Given that t^2 is nonzero, we may divide by t^2 to obtain an equation of the form

$$y'' + \frac{1}{t}y' + \frac{1}{t^2}y = \frac{\ln|t|}{t^2}.$$

Each function $p(t) = t^{-1}$, $q(t) = t^{-2}$, and $g(t) = t^{-2}\ln|t|$ is continuous on its domain, hence the open interval of maximum length for which the initial value problem admits a solution is $(0, \infty)$. ◇

Even though we are not in a position to prove the Fundamental Theorem of Second Order Linear Ordinary Differential Equations, we may still enjoy a discussion of some of the facets of the proof. Certainly, if we know that a differential equation admits a solution (or if we assume that it does), then it is natural to ask about "how many" solutions there are; this is especially tractable in case we are dealing with a homogeneous second order linear ordinary differential equation

$$y'' + p(t)y' + q(t)y = 0. \tag{2.2.2}$$

Proposition 2.2.4 (Principle of Superposition)**.** *Given any solutions ϕ_1 and ϕ_2 of the equation*

$$y'' + p(t)y' + q(t)y = 0,$$

we have that $c_1\phi_1(t) + c_2\phi_2(t)$ is a solution of this equation for any real numbers c_1 and c_2.

Proof. We leave the details of the proof to the reader; however, we note that it is simple to verify that $(c_1\phi_1 + c_2\phi_2)'' + p(t)(c_1\phi_1 + c_2\phi_2)' + q(t)(c_1\phi_1 + c_2\phi_2) = 0$ by the linearity of the derivative. □

Essentially, the Principle of Superposition guarantees that for any homogeneous second order linear ordinary differential equation that admits a solution, there are infinitely many solutions (one for each real number). Conversely, we may ask whether every solution of the initial value problem in Equation (2.2.1) has the form $c_1\phi_1(t) + c_2\phi_2(t)$. Of course, in the first place, we must have that

$$\begin{cases} c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0 \text{ and} \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_0'. \end{cases}$$

Converting this system of linear equations into a matrix equation, we find that

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

admits a unique solution if and only if the matrix involving the real functions ϕ_1 and ϕ_2 is invertible. Consequently, for any real univariate functions $\phi_1(t)$ and $\phi_2(t)$, we define the **Wronskian matrix**

$$\mathscr{W}(\phi_1, \phi_2)(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{bmatrix}$$

over the ring of continuously differentiable univariate functions; the **Wronskian determinant** is

$$W(\phi_1, \phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t).$$

Crucially, we note that for any real univariate functions $\phi_1(t)$ and $\phi_2(t)$, the Wronskian determinant $W(\phi_1, \phi_2)(t)$ is a real univariate function of t, so we may evaluate $W(\phi_1, \phi_2)(t)$ at a real number t_0.

Example 2.2.5. Construct the Wronskian matrix of the following pair of continuously differentiable univariate functions; then, compute the Wronskian determinant.

$$\phi_1(t) = t^2 \qquad \phi_2(t) = \ln(t)$$

Solution. We have that $\phi_1'(t) = 2t$ and $\phi_2'(t) = t^{-1}$, hence the Wronskian matrix is given as follows.

$$\mathscr{W}[t^2, \ln(t)] = \begin{bmatrix} t^2 & \ln(t) \\ 2t & \frac{1}{t} \end{bmatrix}$$

Considering that the Wronskian determinant is the difference $\phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)$ of the product of the diagonal and antidiagonal elements of the Wronskian matrix, we find that

$$W[t^2, \ln(t)] = t - 2t\ln(t) = t[1 - 2\ln(t)]$$

for all real numbers $t > 0$. Be sure to note that the domain of $W(t)$ is the domain of $\phi_2(t)$. Even more, observe that $W(t)$ is zero if and only if $\ln(t) = \frac{1}{2}$ if and only if $t = \sqrt{e}$. ⋄

Example 2.2.6. Construct the Wronskian matrix of the following pair of continuously differentiable univariate functions; then, compute the Wronskian determinant.

$$\phi_1(t) = e^t \qquad \phi_2(t) = e^{-t}$$

Solution. We have that $\phi_1'(t) = e^t$ and $\phi_2'(t) = -e^{-t}$, hence the Wronskian matrix is given as follows.

$$\mathscr{W}[e^t, e^{-t}] = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}$$

Considering that the Wronskian determinant is the difference $\phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)$ of the product of the diagonal and antidiagonal elements of the Wronskian matrix, we find that

$$W[e^t, e^{-t}] = e^t(-e^{-t}) - e^{-t}e^t = -1 - 1 = -2$$

for all real numbers t. Be sure to note that the domain of $W(t)$ is the common domain of $\phi_1(t)$ and $\phi_2(t)$. Even more, the Wronskian determinant of e^t and e^{-t} is nonzero for all real numbers t. ⋄

By the discussion preceding Examples 2.2.5 and 2.2.6, we obtain the following important fact.

Proposition 2.2.7. *Consider any initial value problem with $y(t_0) = y_0$, $y'(t_0) = y_0'$, and*

$$y'' + p(t)y' + q(t)y = 0$$

for some real functions $p(t)$ and $q(t)$ that are continuous on an open interval I that contains the real number t_0. Given any real solutions $\phi_1(t)$ and $\phi_2(t)$ of the initial value problem, there exist real numbers c_1 and c_2 such that $c_1\phi_1(t) + c_2\phi_2(t)$ satisfies the initial value problem if and only if

$$W(\phi_1, \phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)$$

is nonzero at t_0 if and only if $\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0)$ is not zero.

Consequently, the Wronskian determines whether a linear combination of solutions of an initial value problem constitute a solution of the initial value problem. Conversely, the Wronskian controls all possible solutions of an initial value problem in the following sense (see [BD09, Theorem 3.2.4]).

Theorem 2.2.8. *Consider any homogeneous second order linear ordinary differential equation*

$$y'' + p(t)y' + q(t)y = 0$$

for some real functions $p(t)$ and $q(t)$ that are continuous on an open interval I. Given any solutions $\phi_1(t)$ and $\phi_2(t)$ of the above equation, every solution of this equation is of the form $c_1\phi_1(t) + c_2\phi_2(t)$ for some real numbers c_1 and c_2 if and only if there exists a real number t_0 in I such that

$$W(\phi_1, \phi_2)(t_0) = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0.$$

Given any solutions $\phi_1(t)$ and $\phi_2(t)$ of Equation (2.2.2) such that $W(\phi_1, \phi_2)(t)$ is nonzero, we say that $\phi_1(t)$ and $\phi_2(t)$ constitute a **fundamental set of solutions** of the differential equation. Even more, these functions induce the **general solution** $c_1\phi_1(t) + c_2\phi_2(t)$ of the differential equation by Theorem 2.2.8. We outline the importance of the Wronskian and its applications as follows.

Example 2.2.9. Consider any homogeneous second order linear ordinary differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Given that $\phi_1(t) = e^{r_1 t}$ and $\phi_2(t) = e^{r_2 t}$ are solutions of the above equation, determine a necessary and sufficient condition under which these functions constitute a fundamental set of solutions.

Solution. By definition of a fundamental set of solutions, we seek all real numbers t for which the Wronskian $W(\phi_1, \phi_2)(t)$ is nonzero. Considering that $\phi_1'(t) = r_1 e^{r_1 t}$ and $\phi_2'(t) = r_2 e^{r_2 t}$, we have that

$$W(\phi_1, \phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t) = r_2 e^{r_1 t} e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t} = (r_2 - r_1)e^{(r_1 + r_2)t}.$$

We note that the exponential function $e^{(r_1+r_2)t}$ is nonzero, hence the Wronskian is nonzero if and only if $r_2 - r_1$ is nonzero. Consequently, the real functions $\phi_1(t)$ and $\phi_2(t)$ constitute a fundamental set of solutions if and only if r_1 and r_2 are distinct. (Compare this result with Algorithm 2.1.1.) $\diamond$

Example 2.2.10. Prove that $\phi_1(t) = \sin(\ln(t))$ and $\phi_2(t) = \cos(\ln(t))$ form a fundamental set of solutions of the following homogeneous second order linear ordinary differential equation.

$$t^2 y'' + ty' + y = 0 \text{ for all real numbers } t > 0$$

Proof. By definition of a fundamental set of solutions, we must first establish that $\phi_1(t)$ and $\phi_2(t)$ are solutions of the given differential equation; then, we must show that the Wronskian $W(\phi_1, \phi_2)(t)$ is nonzero. Both of these ends are readily achieved by taking first and second derivatives.

$$\phi_1'(t) = \frac{1}{t}\cos(\ln(t)) \qquad \phi_2'(t) = -\frac{1}{t}\sin(\ln(t))$$

$$\phi_1''(t) = -\frac{1}{t^2}\sin(\ln(t)) - \frac{1}{t^2}\cos(\ln(t)) \qquad \phi_2''(t) = -\frac{1}{t^2}\cos(\ln(t)) + \frac{1}{t^2}\sin(\ln(t))$$

Consequently, we find that $\phi_1(t)$ and $\phi_2(t)$ are solutions to the given differential equation.

$$\begin{aligned} t^2\phi_1'' + t\phi_1' + \phi_2 &= -\sin(\ln(t)) - \cos(\ln(t)) + \cos(\ln(t)) + \sin(\ln(t)) = 0 \\ t^2\phi_2'' + t\phi_2' + \phi_2 &= -\cos(\ln(t)) + \sin(\ln(t)) - \sin(\ln(t)) + \cos(\ln(t)) = 0 \end{aligned}$$

Last, we compute the Wronskian of the pair of functions as follows.

$$W(\phi_1, \phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t) = -\frac{1}{t}\sin^2(t) - \frac{1}{t}\cos^2(t) = -\frac{1}{t}$$

Considering that the Wronskian is nonzero for all real numbers $t > 0$, we conclude that the given functions form a fundamental set of solutions on the open interval $(0, \infty)$. $\square$

We have seen in Proposition 2.2.7 and Theorem 2.2.8 that the Wronskian of a pair of solutions of a homogeneous second order linear ordinary differential equation determines the existence and uniqueness of solutions of the differential equation. Explicitly, every solution can be obtained as a linear combination of a given pair of solutions to the differential equation if and only if the Wronskian of the solutions is nonzero at a point. Even more, a pair of solutions constitute a fundamental set of solutions if and only if their Wronskian is nonzero on the interval of definition of the differential equation. Consequently, it is natural to wonder if every homogeneous second order linear ordinary differential equation admits a fundamental set of solutions: indeed, if this were true, then we would be able to write down the solutions of such an ordinary differential equation as soon as we obtain a fundamental set of solutions. We provide an affirmative answer in the form of the following.

Theorem 2.2.11 (Existence of Fundamental Solutions of a Homogeneous Second Order Linear Ordinary Differential Equation)**.** *Consider any homogeneous second order linear differential equation*

$$y'' + p(t)y' + q(t)y = 0$$

for some real functions $p(t)$ and $q(t)$ that are continuous on an open interval I. Given any point t_0 in I, there exist unique real functions $\phi_1(t)$ and $\phi_2(t)$ such that the following conditions hold.

$$\phi_1(t_0) = 1 \qquad \phi_2(t_0) = 0$$
$$\phi_1'(t_0) = 0 \qquad \phi_2'(t_0) = 1$$

Consequently, the differential equation admits a fundamental set of solutions $\phi_1(t)$ and $\phi_2(t)$.

Example 2.2.12. Construct the fundamental set of solutions of the following differential equation that satisfy the conditions guaranteed by the Existence of Fundamental Solutions of a Homogeneous Second Order Linear Ordinary Differential Equation Theorem for the real number $t_0 = 0$.

$$y'' - 9y = 0$$

Solution. We begin by determining a fundamental set of solutions for the differential equation; then, we may modify our solutions so that they satisfy the conditions of the theorem. Considering that the characteristic equation here satisfies that $r^2 - 9 = 0$, the roots of the characteristic polynomial are $r_1 = 3$ and $r_2 = -3$. By Algorithm 2.1.1, we obtain the solutions $\phi_1(t) = e^{3t}$ and $\phi_2(t) = e^{-3t}$. Example 2.2.9 ensures that these form a fundamental set of solutions of the differential equation; however, in view of the fact that $\phi_1'(t) = 3e^{3t}$ so that $\phi_1'(0) = 3$, we must modify these solutions to obtain the desired solutions. Consider the real function $\phi(t) = c_1e^{3t} + c_2e^{-3t}$ for some real numbers c_1 and c_2 that depend on the initial conditions determined by the following system of equations.

$$\begin{cases} c_1 + c_2 = \phi(0) = 1 \\ 3c_1 - 3c_2 = \phi'(0) = 0 \end{cases}$$

We conclude that $c_1 = c_2 = \frac{1}{2}$, hence $\phi_1(t) = \frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t}$ is a solution of the above equation by the Principle of Superposition. We obtain the second solution in a similar manner as follows.

$$\begin{cases} c_1 + c_2 = \phi(0) = 0 \\ 3c_1 - 3c_2 = \phi'(0) = 1 \end{cases}$$

We find that $c_1 = \frac{1}{6}$ and $c_2 = -\frac{1}{6}$, hence $\phi_2(t) = \frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}$ is a solution of the above equation. Last, we verify that these solutions form a fundamental set of solutions by computing the Wronskian.

$$\begin{aligned}
W(\phi_1, \phi_2)(t) &= \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t) \\
&= \left(\frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t}\right)\left(\frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t}\right) - \left(\frac{3}{2}e^{3t} - \frac{3}{2}e^{-3t}\right)\left(\frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}\right) \\
&= \frac{1}{4}e^{6t} + \frac{1}{2} + \frac{1}{4}e^{-6t} - \frac{1}{4}e^{6t} + \frac{1}{2} - \frac{1}{4}e^{-6t} = 1
\end{aligned}$$

Considering that the Wronskian is identically 1 (and therefore nonzero), we conclude that Theorem 2.2.11 is witnessed by the fundamental solutions $\phi_1(t) = \frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t}$ and $\phi_2(t) = \frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}$. ◇

Even if the solutions of a homogeneous second order linear ordinary differential equation are not known, the following theorem of Niels Henrik Abel ensures that it is possible to determine the Wronskian of any pair of solutions. Crucially, Abel's Theorem demonstrates two important points.

1.) The Wronskian $W(\phi_1, \phi_2)(t)$ of any pair of fundamental solutions $\phi_1(t)$ and $\phi_2(t)$ of a homogeneous second order linear ordinary differential equation is unique (up to a constant that depends only on the underlying solutions $\phi_1(t)$ and $\phi_2(t)$ rather than as a function of t).

2.) Either the Wronskian $W(\phi_1, \phi_2)(t)$ of any pair of solutions $\phi_1(t)$ and $\phi_2(t)$ of a homogeneous second order linear ordinary differential equation is identically zero or it is nonzero on the interval of definition of the differential equation. Even more, this can be readily determined by evaluating $W(\phi_1, \phi_2)(t)$ at any conveniently chosen real number t_0.

Theorem 2.2.13 (Abel's Theorem)**.** *Consider any homogeneous second order linear equation*

$$y'' + p(t)y' + q(t)y = 0$$

for some real functions $p(t)$ and $q(t)$ that are continuous on an open interval I. Given any solutions $\phi_1(t)$ and $\phi_2(t)$ of the above differential equation, there exists a real number C depending only on the functions ϕ_1 and ϕ_2 (and not on t) such that the Wronskian of ϕ_1 and ϕ_2 satisfies that

$$W(\phi_1, \phi_2)(t) = C\exp\left(-\int p(t)\,dt\right).$$

Even more, $W(\phi_1, \phi_2)(t)$ is identically zero if and only if $C = 0$; otherwise, it is nonzero on I.

Proof. Considering the brilliance and simplicity of Abel's proof, we provide the argument as follows. By hypothesis that $\phi_1(t)$ and $\phi_2(t)$ are solutions of the above differential equation, it follows that

$$\begin{aligned}
\phi_1'' + p(t)\phi_1' + q(t)\phi_1 &= 0 \text{ and} \\
\phi_2'' + p(t)\phi_2' + q(t)\phi_2 &= 0.
\end{aligned}$$

By multiplying the first equation by $-\phi_2$, multiplying the second equation by ϕ_1, and adding the resulting equations, we obtain the following homogeneous differential equation.

$$(\phi_1\phi_2'' - \phi_1''\phi_2) + p(t)(\phi_1\phi_2' - \phi_1'\phi_2) = 0$$

We note that $W(\phi_1, \phi_2)(t) = (\phi_1 \phi_2' - \phi_1' \phi_2)(t)$, hence the Product Rule for Derivatives yields that

$$W'(\phi_1, \phi_2) = \phi_1 \phi_2'' + \phi_1' \phi_2' - \phi_1' \phi_2' - \phi_1'' \phi_2 = \phi_1 \phi_2'' - \phi_1'' \phi_2.$$

Consequently, we may recognize the above as a first order linear ordinary differential equation

$$W' + p(t)W = 0.$$

We can easily solve this equation by isolating the terms involving W and the terms involving t.

$$\ln|W| = \int \frac{W'}{W} \, dt = -\int p(t) \, dt$$

We conclude that $W(\phi_1, \phi_2)(t) = C \exp\bigl(-\int p(t) \, dt\bigr)$ for some real number C, as desired. Even more, the constant C depends only on $\phi_1(t)$ and $\phi_2(t)$ and not on t: indeed, observe that $\ln|W|$ depends only on the antiderivative of $p(t)$, so we may appeal to the Fundamental Theorem of Calculus. □

Example 2.2.14. Compute the Wronskian of any pair of solutions of the following equation.

$$t^2 y'' - t(t+2)y' + (t+2)y = 0$$

Solution. Observe that for any nonzero real number t, we may reduce the above equation to

$$y'' - \left(1 + \frac{2}{t}\right) y' + \left(\frac{1}{t} + \frac{2}{t^2}\right) y = 0.$$

Consequently, according to Abel's Theorem with $p(t) = -\bigl(1 + \frac{2}{t}\bigr)$, we have that

$$W(\phi_1, \phi_2)(t) = C \exp\left(-\int p(t) \, dt\right) = C \exp\left[\int \left(1 + \frac{2}{t}\right) dt\right] = C \exp\left(t + 2\ln|t|\right) = Ct^2 e^t$$

for any pair of solutions $\phi_1(t)$ and $\phi_2(t)$ of the above equation. ◇

We conclude this section with the following algorithm summarizing the use of the Wronskian.

Remark 2.2.15 (Using the Wronskian)**.** Consider any ordinary differential equation of the form

$$y'' + p(t)y' + q(t)y = 0$$

for some real functions $p(t)$ and $q(t)$ that are continuous on some open interval I. Carry out the following steps to determine the general solution of the above equation.

1.) Begin with two real univariate functions $\phi_1(t)$ and $\phi_2(t)$ that satisfy the differential equation on I. Often in practice, this is difficult (or even impossible) to achieve.

2.) Granted that the solutions $\phi_1(t)$ and $\phi_2(t)$ above can be determined and written down in a closed form using elementary functions, compute the Wronskian $W(\phi_1, \phi_2)(t)$.

3.) If $W(\phi_1, \phi_2)(t)$ is nonzero for some real number t_0 in I, then every solution of the differential equation is of the form $c_1\phi_1(t) + c_2\phi_2(t)$ for some real numbers c_1 and c_2. Even more, this ensures $W(\phi_1, \phi_2)(t)$ is nonzero on I, so $\phi_1(t)$ and $\phi_2(t)$ form a fundamental set of solutions.

4.) Last, if we prescribe initial conditions $y(0) = y_0$ and $y'(0) = y_0'$, then we may uniquely determine the constants c_1 and c_2 from the previous step by solving a system of linear equations.

2.3 Complex Roots of the Characteristic Equation

Consider any homogeneous second order linear ordinary differential equation of the form

$$ay'' + by' + c = 0 \tag{2.3.1}$$

for some real numbers a, b, and c such that a is nonzero. Back in Section 2.1, we determined that in order to solve the above differential equation, it suffices to consider the characteristic equation

$$ar^2 + br + c = 0. \tag{2.3.2}$$

Explicitly, we found that if Equation (2.3.2) admits distinct real roots r_1 and r_2, then for any real numbers c_1 and c_2, we have that $\phi(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ constitutes a solution of Equation (2.3.1). By Example 2.2.9, it follows that this is the unique general solution of this differential equation, hence we have completely solved the second order ordinary differential equation (2.3.1) in this case.

Unfortunately, we are not off the hook here: indeed, it is entirely possible that the **discriminant** $b^2 - 4ac$ of the characteristic polynomial $ar^2 + br + c$ is negative. By the Quadratic Formula, in this case, the distinct non-real complex roots of the characteristic polynomial are given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}.$$

By denoting $\alpha = -\frac{b}{2a}$ and $\beta = \frac{1}{2a}|b^2 - 4ac|$, the roots of the characteristic polynomial are complex conjugates $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$. By analogy to the solution outlined in Section 2.1, we ought to consider functions of the form $\phi_1(t) = e^{(\alpha + \beta i)t}$ and $\phi_2(t) = e^{(\alpha - \beta i)t}$ as we begin our search for a general solution. But how do we deal with an exponential function with a complex exponent?

Theorem 2.3.1 (Euler's Formula)**.** *Given any real number t, we have that* $e^{it} = \cos(t) + i\sin(t)$.

Proof. Consider the power series expansion of e^t below.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

By the Ratio Test, the above power series is valid for all real numbers t. Even more, if we assume that it is valid to substitute it into the above expression, then we obtain a formal power series

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}.$$

Considering that $i^2 = -1$, it can be shown that $i^3 = -i$ and $i^4 = 1$ so that

$$e^{it} = \sum_{n=0}^{\infty} \frac{i^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \cos(t) + i\sin(t). \qquad \square$$

On its face, the above proof merely suggests the identity known as **Euler's Formula**: indeed, the rigorous verification of this fact is beyond the scope of this course; however, the intuition outlined in the proof leads us to define the univariate exponential function of a complex exponent

$$e^{it} = \cos(t) + i\sin(t). \tag{2.3.3}$$

Considering that $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$, we ought to define

$$e^{-it} = \cos(t) - i\sin(t). \tag{2.3.4}$$

Likewise, if we replace t by any real multiple Ct of t, we should find that

$$e^{iCt} = \cos(Ct) + i\sin(Ct). \tag{2.3.5}$$

Even more, the exponential function of a complex exponent should obey the usual laws of exponentiation. Consequently, we must define complex exponentiation according to the following rule.

$$e^{(\alpha+\beta i)t} = e^{\alpha t}e^{i\beta t} \tag{2.3.6}$$

Combined with Equation (2.3.5), this yields the definition of the complex exponential function.

$$e^{(\alpha+\beta i)t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t) \tag{2.3.7}$$

Considering this complex function in terms of its real and complex components, it follows that

$$\frac{d}{dt}e^{rt} = re^{rt} \tag{2.3.8}$$

for any complex exponent $r = \alpha + \beta i$. Consequently, the complex exponential function behaves similarly to the real exponential function — and that is exactly how we have devised it.

Coming back to the solutions of the ordinary differential equation (2.3.1) in the case that $b^2 - 4ac$ is negative, we recall that the roots of the characteristic equation (2.3.2) are the complex conjugates $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ for some real numbers α and β. By the exposition in the second paragraph of this section, we found that the general solution of this equation ought to satisfy that $\phi(t) = c_1\phi_1(t) + c_2\phi_2(t)$ for any real numbers c_1 and c_2 and the complex exponential functions

$$\begin{aligned}\phi_1(t) &= e^{(\alpha+\beta i)t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t) \text{ and}\\ \phi_2(t) &= e^{(\alpha-\beta i)t} = e^{\alpha t}\cos(\beta t) - ie^{\alpha t}\sin(\beta t).\end{aligned}$$

Explicitly, the simplification goes through Equations (2.3.4) and (2.3.7). Even more, we have that

$$\begin{aligned}W(\phi_1, \phi_2)(t) &= \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)\\ &= (\alpha - \beta i)e^{(\alpha+\beta i)t}e^{(\alpha-\beta i)t} - (\alpha + \beta i)e^{(\alpha+\beta i)t}e^{(\alpha-\beta i)t}\\ &= (\alpha - \beta i)e^{2\alpha t} - (\alpha + \beta i)e^{2\alpha t}\\ &= -\beta ie^{2\alpha t}.\end{aligned}$$

Considering that this complex function is nonzero so long as β is nonzero, we may be inclined to report that $\phi_1(t)$ and $\phi_2(t)$ form a fundamental set of solutions for the given differential equation (2.3.1); however, we do not at present have a notion of complex solutions of a differential equation.

We seek to this end to "make real" the strictly complex functions $\phi_1(t)$ and $\phi_2(t)$. By the Principle of Superposition, any linear combination of solutions of a differential equation is again a solution of the differential equation. Consequently, we may consider the real univariate functions

$$\gamma_1(t) = \frac{\phi_1(t) + \phi_2(t)}{2} = e^{\alpha t}\cos(\beta t) \text{ and } \gamma_2(t) = \frac{\phi_1(t) - \phi_2(t)}{2i} = e^{\alpha t}\sin(\beta t). \tag{2.3.9}$$

Bearing in mind that our original aim was to construct a fundamental set of solutions of the ordinary differential equation (2.3.1) in the case that β is a nonzero real number, it suffices to prove that (1.) $\gamma_1(t)$ and $\gamma_2(t)$ satisfy the specified differential equation and (2.) their Wronskian $W(\gamma_1, \gamma_2)(t)$ is nonzero. We may then appeal to Theorem 2.2.8 and Abel's Theorem to conclude the desired result. We omit the details of the first point, but we note that the derivatives of γ_1 and γ_2 satisfy that

$$\gamma_1'(t) = -\beta e^{\alpha t}\sin(\beta t) + \alpha e^{\alpha t}\cos(\beta t) = e^{\alpha t}[\alpha\cos(\beta t) - \beta\sin(\beta t)] \text{ and}$$
$$\gamma_2'(t) = \beta e^{\alpha t}\cos(\beta t) + \alpha e^{\alpha t}\sin(\beta t) = e^{\alpha t}[\alpha\sin(\beta t) + \beta\cos(\beta t)], \text{ hence we have that}$$

$$W(\gamma_1, \gamma_2)(t) = e^{2\alpha t}[\alpha\sin(\beta t)\cos(\beta t) + \beta\cos^2(\beta t) - \alpha\sin(\beta t)\cos(\beta t) + \beta\sin^2(\beta t)] = \beta e^{2\alpha t}.$$

Once again, by assumption that β is nonzero, it follows that $W(\gamma_1, \gamma_2)(t)$ is nonzero, as desired.

Algorithm 2.3.2 (Solutions of Homogeneous Second Order Linear Ordinary Differential Equation with Constant Coefficients II)**.** Given any real univariate function $y = f(t)$ and any real numbers a, b, and c such that a is nonzero, consider the homogeneous second order linear differential equation

$$ay'' + by' + c = 0.$$

Carry out the following steps to determine the general solution $y = \phi(t)$.

1.) Compute the roots of the characteristic equation $ar^2 + br + c = 0$.

2.) Provided that the above characteristic equation admits two distinct real solutions r_1 and r_2, the general solution of the above differential equation is given by $\phi(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

3.) Otherwise, if the characteristic equation admits complex solutions $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, the general solution of the differential equation is given by $\phi(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$.

4.) Last, if the characteristic equation has two identical real solutions, further analysis is required.

Given any initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$,

4.) solve the following system of equations to obtain the solution of the initial value problem.

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases}$$

Example 2.3.3. Construct the solution to the initial value problem with $y(0) = 1$, $y'(0) = 1$, and

$$y'' + y = 0.$$

Solution. Observe that the discriminant of the characteristic equation $r^2 + 1 = 0$ is negative, hence there two complex conjugate solutions $r_1 = i$ and $r_2 = -i$ so that $\alpha = 0$ and $\beta = 1$. We obtain the general solution of the above differential equation as $y(t) = c_1\cos(t) + c_2\sin(t)$; in order to obtain the particular solution of the given initial value problem, we solve the system of equations below.

$$\begin{cases} c_1 = y(0) = 1 \\ c_2 = y'(0) = 1 \end{cases}$$

Consequently, the particular solution we seek is simply $y(t) = \sin(t) + \cos(t)$. ⋄

Example 2.3.4. Construct the solution to the initial value problem with $y(0) = 0$, $y'(0) = 1$, and

$$y'' - 6y' + 13y = 0.$$

Solution. By the Quadratic Formula, the roots of the characteristic equation $r^2 - 6r + 13 = 0$ satisfy

$$r = \frac{6 \pm \sqrt{6^2 - 4(1)(13)}}{2} = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

so that $\alpha = 3$ and $\beta = 2$. We obtain the general solution of the above differential equation as $y(t) = c_1 e^{3t} \cos(2t) + c_2 e^{3t} \sin(2t)$ with $y'(t) = 3y(t) - 2c_1 e^{3t} \sin(2t) + 2c_2 e^{3t} \cos(2t)$ by the Product Rule and Chain Rule. We solve the system of equations below to complete the initial value problem.

$$\begin{cases} c_1 = y(0) = 0 \\ 2c_2 = y'(0) = 1 \end{cases}$$

Consequently, the particular solution we seek is simply $y(t) = \frac{1}{2} e^{3t} \sin(2t)$. ⋄

Example 2.3.5. Construct the solution to the initial value problem with $y(0) = -1$, $y'(0) = 0$, and

$$4y'' - 4y' + 5y = 0.$$

Solution. Once again, the roots of the characteristic equation $4r^2 - 4r + 5 = 0$ are given by

$$r = \frac{4 \pm \sqrt{4^2 - 4(4)(5)}}{8} = \frac{4 \pm \sqrt{16 - 80}}{8} = \frac{4 \pm \sqrt{-64}}{8} = \frac{1}{2} \pm i,$$

and we find that $\alpha = \frac{1}{2}$ and $\beta = 1$. We obtain the general solution of the above differential equation as $y(t) = c_1 e^{t/2} \cos(t) + c_2 e^{t/2} \sin(t)$ with $y'(t) = \frac{1}{2} y(t) - c_1 e^{t/2} \sin(t) + c_2 e^{t/2} \cos(t)$ by the Product Rule and Chain Rule. We solve the system of equations below to complete the initial value problem.

$$\begin{cases} c_1 = y(0) = -1 \\ -\frac{1}{2} + c_2 = y'(0) = 0 \end{cases}$$

Consequently, the particular solution we seek is simply $y(t) = -e^{t/2} \cos(t) + \frac{1}{2} e^{t/2} \sin(t)$. ⋄

2.4 Repeated Roots of the Characteristic Equation

Consider once again any homogeneous second order linear ordinary differential equation of the form

$$ay'' + by' + c = 0 \tag{2.4.1}$$

for some real numbers a, b, and c such that a is nonzero. By the technique laid out in Section 2.1, the above differential equation can be solved by determining the roots of the characteristic equation

$$ar^2 + br + c = 0. \tag{2.4.2}$$

Explicitly, if Equation (2.4.2) admits distinct real roots r_1 and r_2, then Example 2.2.9 and direct verification ensure that $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is the unique general solution of this differential equation for any real numbers c_1 and c_2. Even more, if Equation (2.4.2) admits distinct conjugate complex roots $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, then the general solution of this differential equation is given by $y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$ by the discussion carried out in Section 2.3.

Last, we turn our attention to the case that there exists but one repeated root of the characteristic equation. By the Quadratic Formula, this occurs precisely when the discriminant $b^2 - 4ac$ of the characteristic equation is zero; in this case, the lone root of the characteristic equation is $r = -b/2a$. Consequently, we are led as in Section 2.1 to the solution $\phi(t) = e^{-bt/2a}$; however, by Theorem 2.2.8, every solution of Equation (2.4.1) is a real linear combination some pair of distinct solutions, so it is necessary to seek a second solution that is not a scalar multiple of $\phi(t)$. We turn our attention to the following method of D'Alembert that can be used in general to construct another solution. Considering that this solution $\gamma(t)$ of Equation (2.4.1) is not a scalar multiple of $\phi(t)$, D'Alembert cleverly notes that $\gamma(t)$ must be the product of $\phi(t)$ and a non-constant function $\sigma(t)$ so that

$$\begin{aligned} \gamma(t) &= \sigma(t)\phi(t), \\ \gamma'(t) &= \sigma(t)\phi'(t) + \sigma'(t)\phi(t), \text{ and} \\ \gamma''(t) &= \sigma(t)\phi''(t) + 2\sigma'(t)\phi'(t) + \sigma''(t)\phi(t). \end{aligned}$$

By assumption that $\gamma(t)$ is a solution of Equation (2.4.1), it follows that

$$a\gamma'' + b\gamma' + c\gamma = 0. \tag{2.4.3}$$

Computing the derivatives of $\phi(t) = e^{-bt/2a}$ yields that $\phi'(t) = -\frac{b}{2a} e^{-bt/2a}$ and $\phi''(t) = \frac{b^2}{4a^2} e^{-bt/2a}$ by the Chain Rule; then, replacing the functions $\phi'(t)$ and $\phi''(t)$ with these explicit expressions in the above equation involving the derivatives of $\gamma(t)$ and simplifying the result, we find that

$$\begin{aligned} \gamma(t) &= \sigma(t)e^{-bt/2a}, \\ \gamma'(t) &= \left[-\frac{b}{2a}\sigma(t) + \sigma'(t)\right] e^{-bt/2a}, \text{ and} \\ \gamma''(t) &= \left[\frac{b^2}{4a^2}\sigma(t) - \frac{b}{a}\sigma'(t) + \sigma''(t)\right] e^{-bt/2a}. \end{aligned}$$

Consequently, Equation (2.4.3) can be simplified to the following differential equation in $\gamma(t)$.

$$\left(\left[\frac{b^2}{4a}\sigma(t) - b\sigma'(t) + a\sigma''(t)\right] + \left[-\frac{b^2}{2a}\sigma(t) + b\sigma'(t)\right] + c\sigma(t)\right) e^{-bt/2a} = 0 \tag{2.4.4}$$

Considering that $e^{-bt/2a}$ is nonzero, it can be cancelled from each side of Equation (2.4.4). Collecting the coefficients of $\sigma(t)$, $\sigma'(t)$, and $\sigma''(t)$, we obtain the following differential equation in $\sigma(t)$.

$$a\sigma''(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)\sigma(t) = 0 \tag{2.4.5}$$

By our initial assumption that $b^2 - 4ac = 0$, it follows that $c = \frac{b^2}{4a}$, hence the term in Equation (2.4.5) that involves $\sigma(t)$ is zero. We arrive at last at the ordinary differential equation $a\sigma''(t) = 0$. Considering that a is nonzero by hypothesis, we must have that $\sigma''(t) = 0$, hence the Fundamental Theorem of Calculus ensures that $\sigma(t) = c_1 + c_2 t$ for some real numbers c_1 and c_2 so that

$$\gamma(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}. \tag{2.4.6}$$

Crucially, we note that $\gamma(t)$ is a linear combination of the solution $\phi_1(t) = e^{-bt/2a}$ and the real univariate function $\phi_2(t) = te^{-bt/2a}$ that is itself a solution of Equation (2.4.1): indeed, by construction, $\gamma(t)$ is a solution of this equation, and $\phi_2(t)$ is obtained from $\gamma(t)$ by setting $c_1 = 0$ and $c_2 = 1$. By the Product Rule, we have that $\phi_2'(t) = \left(1 - \frac{b}{2a}t\right)e^{-bt/2a}$, hence the Wronskian of $\phi_1(t)$ and $\phi_2(t)$ is

$$W(\phi_1, \phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t) = \left(1 - \frac{b}{2a}t\right)e^{-bt/a} + \frac{b}{2a}e^{-bt/a} = e^{-bt/a}.$$

Considering that the real univariate exponential function is nonzero, we conclude that the Wronskian is nonzero, hence $\phi_1(t)$ and $\phi_2(t)$ form a fundamental set of solutions of Equation (2.4.1).

Algorithm 2.4.1 (Solutions of Homogeneous Second Order Linear Ordinary Differential Equation with Constant Coefficients III)**.** Given any real univariate function $y = f(t)$ and any real numbers a, b, and c such that a is nonzero, consider the homogeneous second order linear differential equation

$$ay'' + by' + c = 0.$$

Carry out the following steps to determine the general solution $y = \phi(t)$.

1.) Compute the roots of the characteristic equation $ar^2 + br + c = 0$.

2.) Provided that the above characteristic equation admits two distinct real solutions r_1 and r_2, the general solution of the above differential equation is given by $\phi(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

3.) Otherwise, if the characteristic equation admits complex solutions $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$, the general solution of the differential equation is given by $\phi(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$.

4.) Last, if the characteristic equation admits two identical real solutions $r_1 = r_2 = -b/2a$, the general solution of the differential equation is given by $\phi(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$.

Given any initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$,

4.) solve the following system of equations to obtain the solution of the initial value problem.

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases}$$

Example 2.4.2. Construct the solution to the initial value problem with $y(0) = 1$, $y'(0) = 2$, and

$$y'' - 6y' + 9y = 0.$$

Solution. Construct the characteristic equation $r^2 - 6r + 9 = 0$ with discriminant $6^2 - 4(1)(9) = 0$. By the Quadratic Formula, there are two identical roots of the characteristic equation $r_1 = r_2 = 3$, hence the general solution of the above differential equation is $y(t) = c_1 e^{3t} + c_2 t e^{3t}$. By the Product Rule and the Chain Rule, it follows that $y'(t) = 3y(t) + c_2 e^{3t}$, hence we obtain a system of equations.

$$\begin{cases} c_1 = y(0) = 1 \\ 3 + c_2 = y'(0) = 2 \end{cases}$$

We conclude that the particular solution to this initial value problem is $y(t) = e^{3t} - te^{3t}$. ⋄

Example 2.4.3. Construct the solution to the initial value problem with $y(0) = 0$, $y'(0) = 2$, and

$$4y'' + 4y' + y = 0.$$

Solution. Construct the characteristic equation $4r^2 + 4r + 1 = 0$ with discriminant $4^2 - 4(4)(1) = 0$. By the Quadratic Formula, there are two identical roots of the characteristic equation $r_1 = r_2 = -\frac{1}{2}$, hence the general solution of the above differential equation is $y(t) = c_1 e^{-t/2} + c_2 t e^{-t/2}$. By the Product Rule and Chain Rule, we have $y'(t) = -\frac{1}{2} y(t) + c_2 e^{-t/2}$. Consider the system of equations.

$$\begin{cases} c_1 = y(0) = 0 \\ c_2 = y'(0) = 2 \end{cases}$$

We conclude that the particular solution to this initial value problem is $y(t) = 2te^{-t/2}$. ⋄

Example 2.4.4. Construct the solution to the initial value problem with $y(0) = 3$, $y'(0) = 0$, and

$$9y'' - 24y' + 16y = 0.$$

Solution. Our characteristic equation $9r^2 - 24r + 16 = 0$ has discriminant $24^2 - 4(9)(16) = 0$. By the Quadratic Formula, there are two identical roots of the characteristic equation $r_1 = r_2 = \frac{4}{3}$, hence the general solution of the above differential equation is $y(t) = c_1 e^{4t/3} + c_2 t e^{4t/3}$. By the Product Rule and Chain Rule, we have that $y'(t) = \frac{4}{3} y(t) + c_2 e^{4t/3}$. We obtain the following equations.

$$\begin{cases} c_1 = y(0) = 3 \\ 4 + c_2 = y'(0) = 0 \end{cases}$$

We conclude that the particular solution to this initial value problem is $y(t) = 3e^{4t/3} - 4te^{4t/3}$. ⋄

Generally, the method of D'Alembert outlined in the context of homogeneous second order linear ordinary differential equations with constant coefficients can be applied to any homogeneous second order linear equation. Explicitly, we will assume that $\phi(t)$ is a nonzero solution of the equation

$$y'' + p(t)y' + q(t)y = 0. \tag{2.4.7}$$

Copying the idea set forth by D'Alembert, suppose that $\sigma(t)$ is a non-constant function for which $\gamma(t) = \sigma(t)\phi(t)$. By the Product Rule, as in the exposition preceding Equation (2.4.3), we have that

$$\begin{aligned} \gamma(t) &= \sigma(t)\phi(t), \\ \gamma'(t) &= \sigma(t)\phi'(t) + \sigma'(t)\phi(t), \text{ and} \\ \gamma''(t) &= \sigma(t)\phi''(t) + 2\sigma'(t)\phi'(t) + \sigma''(t)\phi(t). \end{aligned}$$

By substituting γ for y in Equation (2.4.7) and simplifying, it follows that

$$\phi\sigma'' + (2\phi' + p\phi)\sigma' + (\phi'' + p\phi' + q\phi)\sigma = 0.$$

Considering that $\phi(t)$ is by assumption a solution of Equation (2.4.7), the coefficient of $\sigma(t)$ in the previous equation is identically zero, hence this line of argument resolves in the following equation.

$$\phi(t)\sigma'' + [2\phi'(t) + p(t)\phi(t)]\sigma' = 0 \tag{2.4.8}$$

Crucially, the result is a homogeneous first order linear ordinary differential equation in $\sigma'(t)$. Even more, Equation (2.4.8) is separable since we may rearrange to isolate the functions $\sigma'(t)$ and $\sigma''(t)$.

Algorithm 2.4.5 (Reduction of Order)**.** Consider any real univariate functions $p(t)$ and $q(t)$ that are continuous and that define a homogeneous second order linear ordinary differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Given that some nonzero solution $y = \phi(t)$ to the above equation is known, carry out the following steps to reduce to a first order equation and determine the general solution of the above equation.

1.) Construct the real univariate function $\sigma(t)\phi(t)$.

2.) Construct the general solution of the following first order linear ordinary differential equation.

$$\phi(t)\sigma'' + [2\phi'(t) + p(t)\phi(t)]\sigma' = 0$$

3.) Compute the Wronskian of $\sigma(t)\phi(t)$ and $\phi(t)$ to ensure a fundamental set of solutions.

Example 2.4.6. Use the Reduction of Order Algorithm to solve the following homogeneous second order linear ordinary differential equation provided that $\phi(t) = \sin(t^2)$ is a solution and $t > 0$.

$$ty'' - y' + 4t^3y = 0$$

Example 2.4.7. Use the Reduction of Order Algorithm to solve the following homogeneous second order linear ordinary differential equation provided that $\phi(t) = t^{-1}$ is a solution and $t > 0$.

$$t^2y'' + 3ty' + y = 0$$

2.5 The Method of Undetermined Coefficients

Consider any non-homogeneous second order linear ordinary differential equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{2.5.1}$$

that is defined for some real univariate functions $p(t)$, $q(t)$, and $g(t)$ that are continuous on an open interval in which $g(t)$ is nonzero for some real number t. We provide first a structure theorem that governs the form of all solutions of Equation (2.5.1). We require only two simple facts to this end.

Proposition 2.5.1 (Reduction of a Non-Homogeneous Equation to the Homogeneous Case)**.** *Consider any non-homogeneous second order linear ordinary differential equation*

$$y'' + p(t)y' + q(t)y = g(t)$$

that is defined for some real univariate functions $p(t)$, $q(t)$, and $g(t)$ that are continuous on an open interval in which $g(t)$ is nonzero for some real number t. Given any pair of solutions $\gamma_1(t)$ and $\gamma_2(t)$ of Equation (2.5.1), their difference $\gamma_1(t) - \gamma_2(t)$ constitutes a solution of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{2.5.2}$$

Consequently, if the real univariate functions $\phi_1(t)$ and $\phi_2(t)$ form a fundamental set of solutions of Equation (2.5.1), then there exist real numbers c_1 and c_2 such that $\gamma_1(t) - \gamma_2(t) = c_1\phi_1(t) + c_2\phi_2(t)$.

Consequently, every non-homogeneous equation (2.5.1) can be reduced to a homogeneous equation (2.5.2) with the same coefficient functions $p(t)$ and $q(t)$. Even more, the following holds.

Theorem 2.5.2 (Fundamental Theorem of Non-Homogeneous Second Order Linear Ordinary Differential Equations)**.** *Consider any non-homogeneous second order linear ordinary differential eq'n*

$$y'' + p(t)y' + q(t)y = g(t)$$

that is defined for some real univariate functions $p(t)$, $q(t)$, and $g(t)$ that are continuous on an open interval on which $g(t)$ is nonzero for some real number t. Each solution of Equation (2.5.1) satisfies

$$y = \phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \gamma(t)$$

for some solution $\gamma(t)$ of the non-homogeneous Equation (2.5.1), some fundamental set of solutions $\phi_1(t)$ and $\phi_2(t)$ of the homogeneous Equation (2.5.2), and some real numbers c_1 and c_2.

We derive the following algorithm for solving Equation (2.5.1) using the previous two facts.

Algorithm 2.5.3 (Solutions of Non-Homogeneous Second Order Linear Equations)**.** Consider any non-homogeneous second order linear ordinary differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

that is defined for some real univariate functions $p(t)$, $q(t)$, and $g(t)$ that are continuous on an open interval in which $g(t)$ is nonzero for some real number t. Carry out the following steps to determine the general solution $y = \phi(t)$ of the above equation (2.5.1).

1.) Construct the general solution $c_1\phi_1(t) + c_2\phi_2(t)$ of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

Call this function the **complementary solution** of the corresponding differential equation.

2.) Construct some solution $\gamma(t)$ of the non-homogeneous second order linear equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Call this function the **particular solution** of the corresponding differential equation.

3.) By the Fundamental Theorem of Non-Homogeneous Second Order Linear Ordinary Differential Equations, the general solution of Equation (2.5.1) is $y = \phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \gamma(t)$.

Unfortunately, we have not yet developed a tried-and-true method for tackling the second step of Algorithm 2.5.3. Bearing this in mind, we turn our attention to resolving this issue throughout the remainder of this section and the next. Out of desire for simplicity (and by analogy to the work we have done in this chapter so far), we will consider first the case of a non-homogeneous second order equation with constant coefficients. Explicitly, we will assume that $y = f(t)$ satisfies that

$$ay'' + by' + c = g(t) \tag{2.5.3}$$

for some real numbers a, b, and c such that a is nonzero and some real univariate function $g(t)$ that is continuous on some open interval in which $g(t)$ is nonzero for some real number t. Considering that $g(t)$ is a linear combination of y, y', and y'', our best guess for a potential solution of Equation (2.5.3) is some scalar multiple of $g(t)$. Consequently, we assume that there exists a real number A for which $\phi(t) = Ag(t)$ is a solution of Equation (2.5.3). By taking the first and second derivatives of $\phi(t)$ and plugging them into Equation (2.5.3), we obtain a linear equation in $g(t)$, $g'(t)$, and $g''(t)$.

$$aAg''(t) + bAg'(t) + cAg(t) = a\phi''(t) + b\phi'(t) + c\phi(t) = g(t)$$

Under "ideal" conditions, this linear equation can be solved using the linear algebra of systems of equations. We refer to this technique in general as the **method of undetermined coefficients** because in order to solve the differential equation, it suffices to solve for the unknown constant A.

Example 2.5.4. Construct a particular solution $\gamma(t)$ of the non-homogeneous equation

$$y'' + 3y' + 2y = 8e^{-3t}.$$

Solution. We will assume that $\gamma(t) = Ae^{-3t}$ satisfies this equation for some real number A so that

$$2Ae^{-3t} = 9Ae^{-3t} - 9Ae^{-3t} + 2Ae^{-3t} = \gamma'' + 3\gamma' + 2\gamma = 8e^{-3t}.$$

Cancelling a factor of the nonzero exponential function e^{-3t} from each side of this equation yields that $2A = 8$ so that $A = 4$. We conclude that $\gamma(t) = 4e^{-3t}$ is a particular solution of the equation. ◇

Example 2.5.5. Construct a particular solution $\gamma(t)$ of the non-homogeneous equation

$$y'' + 3y' + 2y = 7e^{-t}.$$

Solution. We will assume that $\gamma(t) = Ae^{-t}$ satisfies this equation for some real number A so that

$$0 = Ae^{-t} - 3Ae^{-t} + 2Ae^{-t} = 7e^{-t}.$$

Clearly, this is a contradiction because the exponential function $7e^{-t}$ is nonzero, hence our hypothesis on the form of the solution $\gamma(t)$ is incorrect; however, if we assume instead that $\gamma(t) = Ate^{-t}$ satisfies the above equation for some real number A, then we obtain another equation

$$Ae^{-t} = (-2Ae^{-t} + Ate^{-t}) + 3(Ae^{-t} - Ate^{-t}) + 2Ate^{-t} = \gamma'' + 3\gamma' + 2\gamma = 7e^{-t}.$$

Cancelling a factor of the nonzero exponential function e^{-t} from each side of this equation yields that $A = 7$ so that $\gamma(t) = 7te^{-t}$ is a particular solution of the above differential equation. ◇

Caution: Example 2.5.5 illustrates that the method of undetermined coefficients may fail at first; however, it is typically possible to repair this failure and ultimately use it to find a particular solution. Explicitly, in the previous example, hindsight suggests that our choice of $\gamma(t) = Ae^{-t}$ was doomed to failure: indeed, the characteristic equation $y'' + 3y' + 2y$ is $r^2 + 3r + 2 = (r + 1)(r + 2)$, hence Algorithm 2.1.1 asserts that the homogeneous equation $y'' + 3y' + 2y = 0$ admits the general solution $c_1 e^{-t} + c_2 e^{-2t}$, so we must avoid linear combinations of e^{-t} and e^{-2t} in our choice of $\gamma(t)$.

Example 2.5.6. Construct a particular solution $\gamma(t)$ of the non-homogeneous equation

$$y'' + 3y' + 2y = \sin(t).$$

Solution. We will assume that $\gamma(t) = A\sin(t)$ satisfies this equation for some real number A so that

$$A\sin(t) + 3A\cos(t) = -A\sin(t) + 3A\cos(t) + 2A\sin(t) = \sin(t).$$

Considering this as a function identity that must be valid for all possible choices of t, it follows that $3A = 0$ by plugging in $t = 0$ and $A = 1$ by plugging in $t = \frac{\pi}{2}$ — a contradiction. Consequently, the particular solution of this equation is not of the form $A\sin(t)$ for any real number A; however, if we assume instead that $\gamma(t) = A\sin(t) + B\cos(t)$ for some real numbers A and B, then

$$[-A\sin(t) - B\cos(t)] + 3[A\cos(t) - B\sin(t)] + 2[A\sin(t) + B\cos(t)] = \sin(t)$$

yields a function identity $(A - 3B)\sin(t) + (3A + B)\cos(t) = \sin(t)$. Once again, by evaluating this equation at $t = 0$ and $t = \frac{\pi}{2}$, we obtain a 2×2 system of linear equations

$$\begin{cases} 3A + B = 0 \\ A - 3B = 1 \end{cases}$$

By solving this system of equations (using any valid method), we find that $A = \frac{1}{10}$ and $B = -\frac{3}{10}$ so that $\gamma(t) = \frac{1}{10}\sin(t) - \frac{3}{10}\cos(t)$ constitutes a particular solution of the given equation. ◇

Caution: Example 2.5.6 once again reveals that the method of undetermined coefficients may require slight modifications. Explicitly, in the previous example, it turned out that $\cos(t)$ appeared in the differential equation, so we assumed that it must appear in the particular solution.

Example 2.5.7. Construct a particular solution $\gamma(t)$ of the non-homogeneous equation

$$y'' + 3y' + 2y = 2te^{-t}.$$

Solution. We will assume that $\gamma(t) = Ate^{-t}$ satisfies this equation for some real number A so that

$$Ae^{-t} = (-2Ae^{-t} + Ate^{-t}) + 3(Ae^{-t} - Ate^{-t}) + 2Ate^{-t} = 2te^{-t}.$$

Cancelling a factor of the nonzero exponential function e^{-t} from each side of this equation yields that $A = 2t$ — a contradiction. Consequently, we will assume that $\gamma(t) = Ate^{-t} + Bt^2e^{-t}$ so that

$$Ae^{-t} + (2Be^{-t} - 4Bte^{-t} + Bt^2e^{-t}) + 3(2Bte^{-t} - Bt^2e^{-t}) + 2Bt^2e^{-t} = 2te^{-t}.$$

Cancelling the nonzero exponential function e^{-t} from both sides of the above equation and simplifying, we find that $2Bt + (A + 2B) = 2t$. Consequently, it follows that $2B = 2$ and $A + 2B = 0$ by comparing coefficients, hence we conclude that $A = -2$, $B = 2$, and $\gamma(t) = -2te^{-t} + 2t^2e^{-t}$. ◇

Example 2.5.8. Construct a particular solution $\gamma(t)$ of the non-homogeneous equation

$$y'' + 3y' + 2y = 8e^{-3t} + 7e^{-t} + \sin(t) + 2te^{-t}.$$

Solution. Each of the Examples 2.5.4, 2.5.5, 2.5.6, and 2.5.7 yield particular solutions of the respective non-homogeneous equations with $8e^{-3t}$, $7e^{-t}$, $\sin(t)$, and $2te^{-t}$. Consequently, by the linearity of the derivative, their sum is a particular solution of the differential equation at hand.

$$\gamma(t) = 4e^{-3t} + 7te^{-t} + \frac{1}{10}\sin(t) - \frac{3}{10}\cos(t) - 2te^{-t} + 2t^2e^{-t} \qquad \diamond$$

Last, we provide the general form for the method of undetermined coefficients in common cases. Curious readers may consult the proof on page 181 of the textbook (cf. [BD09, Table 3.5.1]).

Formula 2.5.9 (Method of Undetermined Coefficients)**.** Consider any non-homogeneous second order linear ordinary differential equation with constant coefficients

$$ay'' + by' + cy = g(t)$$

for some real numbers a, b, and c such that a is nonzero and some real univariate function $g(t)$ that is continuous on some open interval in which $g(t)$ is nonzero for some real number t.

1.) If $g(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$ for some real numbers $a_0, a_1, \ldots, a_n$, consider the multiplicity s of the root $r = 0$ of the characteristic equation. We obtain the general solution of Equation (2.5.3) as $\gamma(t) = t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$ for some real numbers $A_0, A_1, \ldots, A_n$.

2.) If $g(t) = (a_0t^n + a_1t^{n-1} + \cdots + a_n)e^{\alpha t}$ for some real numbers $a_0, a_1, \ldots, a_n, \alpha$, consider the multiplicity s of the root $r = \alpha$ of the characteristic equation. We obtain the general solution of Equation (2.5.3) as $\gamma(t) = t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$ for some real numbers $A_0, A_1, \ldots, A_n$.

3.) If $g(t) = (a_0t^n + a_1t^{n-1} + \cdots + a_n)e^{\alpha t}\sin(\beta t)$ or $g(t) = (a_0t^n + a_1t^{n-1} + \cdots + a_n)e^{\alpha t}\cos(\beta t)$ for some real numbers $a_0, a_1, \ldots, a_n, \alpha, \beta$, consider the multiplicity s of the root $r = \alpha + \beta i$ of the characteristic equation. We obtain the general solution of Equation (2.5.3) as

$$\gamma(t) = t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\sin(\beta t) + t^s(B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\cos(\beta t)$$

for some real numbers $A_0, A_1, \ldots, A_n, B_0, B_1, \ldots, B_n$.

Even more, the method of undetermined coefficients is self-correcting: if too little is assumed about the general solution $\gamma(t)$, then a contradiction is obtained. Conversely, if too much is assumed about the general solution $\gamma(t)$, then some coefficients of the general solution vanish without harm.

2.6 Variation of Parameters

Combined with the Method of Undetermined Coefficients, the technique of variation of parameters proves a formidable method for solving non-homogeneous second order linear ordinary differential equations: indeed, by the end of this section, we will arrive at a formula for the general solution of any non-homogeneous second order linear ordinary differential equation that depends only on a fundamental set of solutions of the corresponding homogeneous equation.

Consider any non-homogeneous second order linear ordinary differential equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{2.6.1}$$

defined for some real univariate functions $p(t)$, $q(t)$, and $g(t)$ that are continuous on an open interval in which $g(t)$ is nonzero for some real number t. By the Fundamental Theorem of Non-Homogeneous Second Order Linear Ordinary Differential Equations, every solution of equation (2.6.1) satisfies

$$y = \phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \gamma(t)$$

for some solution $\gamma(t)$ of the non-homogeneous Equation (2.6.1), some fundamental set of solutions $\phi_1(t)$ and $\phi_2(t)$ of the corresponding homogeneous equation, and some real numbers c_1 and c_2. Consequently, if the general solution $c_1\phi_1(t) + c_2\phi_2(t)$ of the corresponding homogeneous equation is known, then $\gamma(t)$ may be determined according to the observation that $\gamma(t)$ must not be a linear combination of $\phi_1(t)$ and $\phi_2(t)$. Crucially, we assume there exist functions $u_1(t)$ and $u_2(t)$ such that

$$\begin{aligned}
\gamma(t) &= u_1(t)\phi_1(t) + u_2(t)\phi_2(t) \text{ so that} \\
\gamma'(t) &= u_1'(t)\phi_1(t) + u_1(t)\phi_1'(t) + u_2'(t)\phi_2(t) + u_2(t)\phi_2'(t) \text{ and} \\
\gamma''(t) &= u_1''(t)\phi_1(t) + 2u_1'(t)\phi_1'(t) + u_1(t)\phi_1''(t) + u_2''(t)\phi_2(t) + 2u_2'(t)\phi_2'(t) + u_2(t)\phi_2''(t).
\end{aligned}$$

Brilliantly, Lagrange observed that we may impose some additional conditions on the first derivatives of the functions $u_1(t)$ and $u_2(t)$. Explicitly, according to Lagrange, we may assume that

$$u_1'(t)\phi_1(t) + u_2'(t)\phi_2(t) = 0. \tag{2.6.2}$$

Consequently, the derivatives of $\gamma(t)$ can be simplified to obtain that

$$\begin{aligned}
\gamma(t) &= u_1(t)\phi_1(t) + u_2(t)\phi_2(t), \\
\gamma'(t) &= u_1(t)\phi_1'(t) + u_2(t)\phi_2'(t), \text{ and} \\
\gamma''(t) &= u_1'(t)\phi_1'(t) + u_1(t)\phi''(t) + u_2'(t)\phi_2'(t) + u_2(t)\phi_2''(t).
\end{aligned}$$

We note that $\gamma(t)$ represents a solution of the non-homogeneous equation (2.6.1) so that

$$\begin{aligned}
g(t) &= \gamma'' + p(t)\gamma' + q(t)\gamma \\
&= u_1'\phi_1' + u_1\phi_1'' + u_2'\phi_2' + u_2\phi_2'' + p(t)(u_1\phi_1' + u_2\phi_2') + q(t)(u_1\phi_1 + u_2\phi_2) \\
&= u_1'\phi_1' + u_2'\phi_2' + [\phi_1'' + p(t)\phi_1' + q(t)\phi_1]u_1 + [\phi_2'' + p(t)\phi_2' + q(t)\phi_2]u_2.
\end{aligned}$$

Considering that $\phi_1(t)$ and $\phi_2(t)$ are by assumption solutions of the corresponding homogeneous equation, the terms in parentheses above are identically zero, hence we obtain a differential equation

$$u_1'(t)\phi_1'(t) + u_2'(t)\phi_2'(t) = g(t). \tag{2.6.3}$$

Equations 2.6.2 and 2.6.3 can be viewed together as a system of algebraic equations that is linear in the functions $u_1'(t)$ and $u_2'(t)$, hence $u_1'(t)$ and $u_2'(t)$ can be obtained by elementary linear algebra: indeed, we must first construct the corresponding matrix equation that represents the system.

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

Considering that $\phi_1(t)$ and $\phi_2(t)$ form a fundamental set of solutions by assumption, the coefficient matrix above is invertible: explicitly, it is none other than the Wronskian matrix $\mathscr{W}(\phi_1, \phi_2)(t)$ whose determinant (i.e., the Wronskian) is nonzero by hypothesis. Consequently, it follows that

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ g(t) \end{bmatrix} = \frac{1}{\phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t)} \begin{bmatrix} \phi_2'(t) & -\phi_2(t) \\ -\phi_1'(t) & \phi_1(t) \end{bmatrix} \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

Computing the matrix product on the right-hand side above and recognizing the denominator of the fraction as the Wronskian of $\phi_1(t)$ and $\phi_2(t)$, we conclude that

$$u_1'(t) = -\frac{\phi_2(t)g(t)}{W(\phi_1, \phi_2)(t)} \text{ and } u_2'(t) = \frac{\phi_1(t)g(t)}{W(\phi_1, \phi_2)(t)}. \tag{2.6.4}$$

Last, computing the antiderivatives of $u_1'(t)$ and $u_2'(t)$ yields the general solution of Equation 2.6.1.

$$y = \phi(t) = c_1\phi_1(t) + c_2\phi_2(t) - \phi_1(t) \int \frac{\phi_2(t)g(t)}{W(\phi_1, \phi_2)(t)}\, dt + \phi_2(t) \int \frac{\phi_1(t)g(t)}{W(\phi_1, \phi_2)(t)}\, dt \tag{2.6.5}$$

Theorem 2.6.1 (Solutions of Second Order Linear Ordinary Differential Equations)**.** *Consider any real functions $p(t)$, $q(t)$, and $g(t)$ that are continuous on an open interval I. Given any fundamental set of solutions $\phi_1(t)$ and $\phi_2(t)$ of the homogeneous second order linear ordinary differential equation*

$$y'' + p(t)y' + q(t)y = 0,$$

the general solution of the corresponding second order linear ordinary differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

can be written in terms of some real number t_0 lying in I and some real numbers c_1 and c_2 as

$$y = \phi(t) = c_1\phi_1(t) + c_2\phi_2(t) - \phi_1(t) \int_{t_0}^{t} \frac{\phi_2(s)g(s)}{W(\phi_1, \phi_2)(s)}\, ds + \phi_2(t) \int_{t_0}^{t} \frac{\phi_1(s)g(s)}{W(\phi_1, \phi_2)(s)}\, ds.$$

Example 2.6.2. Construct the solution to the initial value problem $y(0) = 1$, $y'(0) = -1$, and

$$y'' - y' - 2y = 2e^{-t}.$$

Solution. By the above Solutions of Second Order Linear Ordinary Differential Equations theorem, it suffices to determine the general solution of the corresponding homogeneous equation

$$y'' - y' - 2y = 0.$$

We note that the characteristic polynomial $r^2 - r - 2$ admits two distinct real roots: indeed, it can be factored as $r^2 - r - 2 = (r + 1)(r - 2)$, hence the general solution of the above homogeneous equation is $c_1e^{-t} + c_2e^{2t}$. Explicitly, it follows that $\phi_1(t) = e^{-t}$ and $\phi_2(t) = e^{2t}$ form a fundamental set of solutions of the homogeneous equation; in fact, their Wronskian is given by

$$W(\phi_1, \phi_2)(t) = \phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t) = e^{-t}(2e^{2t}) - (-e^{-t})e^{2t} = 3e^t.$$

Consequently, the general solution of the non-homogeneous equation at hand is determined by

$$\int_{t_0}^t \frac{\phi_2(s)g(s)}{W(\phi_1,\phi_2)(s)}\,ds = \int_0^t \frac{2e^s}{3e^s}\,ds = \frac{2}{3}t \text{ and}$$

$$\int_{t_0}^t \frac{\phi_1(s)g(s)}{W(\phi_1,\phi_2)(s)}\,ds = \int_0^t \frac{2e^{-2s}}{3e^s}\,ds = \frac{2}{3}\int_0^t e^{-3s}\,ds = -\frac{2}{9}e^{-3t} + \frac{2}{9}.$$

We note that our choice of lower bound t_0 was made entirely in service of simplifying the definite integrals at hand. We are at liberty to make this choice because the differential equation is defined for all real numbers. We conclude that the general solution of the non-homogeneous equation is

$$y = \phi(t) = c_1e^{-t} + c_2e^{2t} - \frac{2}{3}te^{-t} + e^{2t}\left(\frac{2}{9} - \frac{2}{9}e^{-3t}\right) = c_1e^{-t} + c_2e^{2t} - \frac{2}{3}te^{-t}.$$

Considering that $y(0) = 1$ and $y'(0) = -1$, we obtain the following system of equations.

$$\begin{cases} c_1 + c_2 = y(0) = 1 \\ -c_1 + 2c_2 - \dfrac{2}{3} = y'(0) = -1 \end{cases}$$

By adding the first and second equations above and simplifying, we find that $3c_2 = \frac{2}{3}$ so that $c_2 = \frac{2}{9}$. Considering that $c_1 = 1 - c_2$ by the first equation above, it follows that $c_1 = \frac{7}{9}$. ◇

Example 2.6.3. Construct the solution to the initial value problem $y(0) = 1$, $y'(0) = -1$, and

$$y'' + 2y' + 2y = te^{-t}.$$

Solution. By the above Solutions of Second Order Linear Ordinary Differential Equations theorem, we must first determine the general solution of the corresponding homogeneous equation

$$y'' + 2y' + 2y = 0.$$

We note that the discriminant of the characteristic polynomial r^2+2r+2 is $2^2-4(1)(2) = -4$, hence the roots of the characteristic equation are $-1 \pm i$. We conclude that the general solution of the homogeneous equation is $c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t)$. Explicitly, it follows that $\phi_1(t) = e^{-t}\cos(t)$ and $\phi_2(t) = e^{-t}\sin(t)$ form a fundamental set of solutions of the homogeneous equation with Wronskian

$$W(\phi_1,\phi_2)(t) = e^{-t}\cos(t)[e^{-t}\cos(t) - e^{-t}\sin(t)] + [e^{-t}\sin(t) + e^{-t}\cos(t)]e^{-t}\sin(t) = e^{-2t}.$$

Consequently, the general solution of the non-homogeneous equation at hand is determined by

$$\int_{t_0}^t \frac{\phi_2(s)g(s)}{W(\phi_1,\phi_2)(s)}\,ds = \int_0^t \frac{se^{-2s}\sin(s)}{e^{-2s}}\,ds = \int_0^t s\sin(s)\,ds = \sin(t) - t\cos(t) \text{ and}$$

$$\int_{t_0}^t \frac{\phi_1(s)g(s)}{W(\phi_1,\phi_2)(s)}\,ds = \int_0^t \frac{se^{-2s}\cos(s)}{e^{-2s}}\,ds = \int_0^t s\cos(s)\,ds = t\sin(t) + \cos(t) - 1.$$

We note that our choice of lower bound t_0 was made entirely in service of simplifying the definite integrals at hand. We are at liberty to make this choice because the differential equation is defined for all real numbers. We conclude that the general solution of the non-homogeneous equation is

$$y = \phi(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) - e^{-t}\cos(t)[\sin(t) - t\cos(t)] + e^{-t}\sin(t)[t\sin(t) + \cos(t) - 1].$$

Cleaning this up a bit by combining like terms and using that $\sin^2(t) + \cos^2(t) = 1$, we find that

$$y = \phi(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) + te^{-t}.$$

Considering that $y(0) = 1$ and $y'(0) = -1$, we obtain the following system of equations.

$$\begin{cases} c_1 = y(0) = 1 \\ -c_1 + c_2 + 1 = y'(0) = -1 \end{cases}$$

We conclude that $c_1 = 1$ and $c_2 = -1$ so that $y = \phi(t) = e^{-t}\cos(t) - e^{-t}\sin(t) + te^{-t}$. ⋄

Example 2.6.4. Construct the solution to the initial value problem $y(0) = 1$, $y'(0) = -1$, and

$$y'' - 2y' + y = \frac{e^t}{1+t^2}.$$

Solution. By the above Solutions of Second Order Linear Ordinary Differential Equations theorem, we must first determine the general solution of the corresponding homogeneous equation

$$y'' - 2y' + y = 0.$$

We note that the characteristic polynomial $r^2 - 2r + 1$ factors as a perfect square trinomial $(r-1)^2$, hence the general solution of the homogeneous equation is $c_1 e^t + c_2 te^t$. Explicitly, it follows that $\phi_1(t) = e^t$ and $\phi_2(t) = te^t$ form a fundamental set of solutions of the homogeneous equation and

$$W(\phi_1, \phi_2)(t) = e^t(te^t + e^t) - e^t(te^t) = e^{2t}.$$

Consequently, the general solution of the non-homogeneous equation at hand is determined by

$$\int_{t_0}^t \frac{\phi_2(s)g(s)}{W(\phi_1,\phi_2)(s)}\,ds = \int_0^t \frac{se^{2s}}{e^{2s}(1+s^2)}\,ds = \int_0^t \frac{s}{1+s^2}\,ds = \frac{1}{2}\ln(1+t^2) \text{ and}$$

$$\int_{t_0}^t \frac{\phi_1(s)g(s)}{W(\phi_1,\phi_2)(s)}\,ds = \int_0^t \frac{e^{2s}}{e^{2s}(1+s^2)}\,ds = \int_0^t \frac{1}{1+s^2}\,ds = \arctan(t).$$

We note that our choice of lower bound t_0 was made entirely in service of simplifying the definite integrals at hand. We are at liberty to make this choice because the differential equation is defined for all real numbers. We conclude that the general solution of the non-homogeneous equation is

$$y = \phi(t) = c_1 e^t + c_2 te^t - \frac{1}{2}e^t\ln(1+t^2) + te^t\arctan(t).$$

Considering that $y(0) = 1$ and $y'(0) = -1$, we obtain the following system of equations.

$$\begin{cases} c_1 = y(0) = 1 \\ c_1 + c_2 = y'(0) = -1 \end{cases}$$

We conclude that $c_1 = 1$ and $c_2 = -2$ so that $y = \phi(t) = e^t - 2te^t - \frac{1}{2}e^t\ln(1+t^2) + te^t\arctan(t)$. ⋄

2.7 Review of Power Series

Like the title suggests, we turn our attention in this section to a brief review of power series. Our aim is to provide a reasonably self-contained exposition on the subject, but the interested reader is encouraged to consult our Calculus II notes for a more in-depth discussion of sequences and series in general. Going forward, we assume a modest familiarity with the typical convergence tests for sequences and series, but embedded links to resources are included throughout this section.

Recall that a nonzero real **polynomial** function of **degree** n is any function of the form

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

for some integer $n \geq 0$ and real numbers $a_0, a_1, \ldots, a_n$ such that a_n is nonzero. We refer to a_i as the **coefficient** of the **monomial** x^i for each integer $0 \leq i \leq n$; the monomials $a_i x^i$ are the **terms** of the polynomial. Using the notion of infinite series, we obtain a generalization of polynomials that allows us to include terms of arbitrarily large degree. Explicitly, we define the **power series**

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots .$$

We refer to the real number c as the **center** of the power series $f(x)$. Under this identification, a polynomial function is simply a power series for which the sequence of coefficients a_n is nonzero for only finitely many integers $n \geq 0$ (i.e., we have that $a_n = 0$ for all sufficiently large integers n).

Convergence of a power series depends not only on its coefficients a_n but also its center c.

Example 2.7.1. Consider the power series centered at $x = 0$ with sequence of coefficients $a_n = n$.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} n x^n$$

Crucially, we note that the value of the function $f(x)$ at $x = a$ is determined by the convergence of the underlying infinite series $f(a)$. Explicitly, if $x = -1$, $x = \frac{1}{2}$, or $x = 1$, the following hold.

$$f(-1) = \sum_{n=0}^{\infty} (-1)^n n \qquad f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{n}{2^n} \qquad f(1) = \sum_{n=0}^{\infty} n$$

By the Divergence Test, it follows that the infinite series corresponding to $f(-1)$ and $f(1)$ diverge. By the Ratio Test, the infinite series corresponding to $f\left(\frac{1}{2}\right)$ converges: indeed, we have that

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Theorem 2.7.2 (Ratio Test)**.** *Given any infinite sequence of real numbers a_n, consider the limit*

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(a.) *If $L < 1$, then the infinite series $\sum_{n=m}^{\infty} a_n$ converges absolutely.*

(b.) *If* $L > 1$, *then the infinite series* $\sum_{n=m}^{\infty} a_n$ *diverges.*

(c.) *If* $L = 1$, *then the Ratio Test is inconclusive: the infinite series* $\sum_{n=m}^{\infty} a_n$ *may diverge.*

Proof. We can easily dispense of the case (c.) that $L = 1$ by considering the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Both of these infinite series satisfy that $L = 1$; however, the former is the famously divergent harmonic series, and the latter converges by the p-Series Test since it is a p-series with $p = 2$.

Likewise, if $L = \infty$, then for all sufficiently large integers n, we have that $|a_{n+1}| > |a_n|$, hence the sequence a_n is eventually increasing without bound in absolute value. By the Divergence Test, we conclude that the infinite series diverges (because its sequence of terms diverges).

We may assume that L is finite. By definition of the limit L, given any real number $\varepsilon > 0$, there exists a positive integer m sufficiently large such that for all integers $n \geq m$, we have that

$$-\varepsilon < \left|\frac{a_{n+1}}{a_n}\right| - L < \varepsilon.$$

By simplifying this inequality, for all integers $n \geq m$, it follows that

$$L - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon.$$

Consider the real numbers $r = L + \varepsilon$ and $s = L - \varepsilon$. We proceed by cases as follows.

(a.) If $L < 1$, we can ensure that $r < 1$ by taking ε to be sufficiently small. Observe that

$$\begin{aligned}
&|a_{m+1}| < |a_M| r, \\
&|a_{m+2}| < |a_{m+1}| r < |a_M| r^2, \\
&|a_{m+3}| < |a_{m+2}| r < |a_{m+1}| r^2 < |a_M| r^3,
\end{aligned}$$

and in general, it holds that $|a_{m+n}| < |a_M| r^n$. Consequently, we have that

$$\sum_{n=m}^{\infty} |a_n| = \sum_{k=0}^{\infty} |a_{m+k}| = \sum_{n=0}^{\infty} |a_{m+n}| < \sum_{n=0}^{\infty} |a_M| r^n = |a_M| \sum_{n=0}^{\infty} r^n.$$

By the Convergence of Geometric Series, the geometric series on the right-hand side converges by hypothesis that $0 \leq L < r < 1$. By the Direct Comparison Test, the series $\sum |a_n|$ converges, hence the series in question converges absolutely by definition of absolute convergence.

(b.) If $L > 1$, we can ensure that $s > 1$ by taking ε to be sufficiently small. By a similar argument as above, it follows that $|a_{m+n}| > |a_M| s^n$. Considering that $s > 1$, it follows that

$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} |a_{m+n}| > \lim_{n \to \infty} |a_M| s^n = |a_M| \lim_{n \to \infty} s^n = \infty.$$

Consequently, the series in question diverges by the Divergence Test. □

Generally, the convergence of a power series can be determined by the Ratio Test as follows.

Theorem 2.7.3 (Convergence of Power Series)**.** *Consider the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

centered at the real number c determined by the infinite sequence of real numbers a_n. There exists a (possibly infinite) real number $\rho \geq 0$ called the **radius of convergence** *of $f(x)$ such that*

(a.) *$f(x)$ converges absolutely for all real numbers x such that $c - \rho < x < c + \rho$ and*

(b.) *$f(x)$ diverges for all real numbers x such that $x > c + \rho$ or $x < c - \rho$.*

We refer to the interval $I = (c - \rho, c + \rho)$ where $f(x)$ converges as the **interval of convergence**. *Even more, the radius of convergence and the interval of convergence satisfy the following properties.*

(a.) *If $\rho = \infty$, then $f(x)$ converges absolutely for all real numbers, i.e., $I = (-\infty, \infty)$.*

(b.) *If $\rho > 0$ is finite, then $f(x)$ may converge or diverge at $x = c - \rho$ and $x = c + \rho$.*

(c.) *If $\rho = 0$, then $f(x)$ diverges for all real numbers $x \neq c$ and $f(x)$ converges for $x = c$.*

Proof. By the Ratio Test, the convergence of the power series $f(x)$ is determined by the following.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n (x-c)^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}(x-c)}{a_n} \right| = |x - c| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Consider the (possibly infinite) real number ρ such that the following equality holds.

$$\frac{1}{\rho} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Certainly, if the above limit tends to 0, then L tends to 0 and ρ tends to infinity. Even more, in this case, we have that $L = 0$, hence the Ratio Test ensures that $f(x)$ converges absolutely for all real numbers x. Conversely, if the above limit tends to infinity, then L tends to infinity and ρ tends to 0, hence the Ratio Test ensures that $f(x)$ diverges for all real numbers $x \neq c$ and $f(x)$ converges for $x = c$. Last, if the above limit is finite, then the real number $\rho \geq 0$ is finite. By the Ratio Test, the power series $f(x)$ converges absolutely if and only if $L < 1$ if and only if

$$\frac{|x - c|}{\rho} < 1$$

if and only if $|x - c| < \rho$ if and only if $-\rho < x - c < \rho$ if and only if $c - \rho < x < c + \rho$. Put another way, we have that $f(x)$ converges absolutely for all real numbers x such that $c - \rho < x < c + \rho$ and $f(x)$ diverges for all real numbers x such that $x > c + \rho$ or $x < c - \rho$. □

Caution: be very careful to note that the Convergence of Power Series theorem does not guarantee anything about the convergence of $f(x)$ when $x = c - \rho$ or $x = c + \rho$ in the case that the radius of convergence ρ is finite and nonzero; rather, we must explicitly test for convergence at these points.

Example 2.7.4. Construct the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. We proceed by the Ratio Test.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right| && \text{(Group like terms.)} \\ &= |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} && \text{(Cancel and factor out constants.)} \\ &= |x| \lim_{n\to\infty} \frac{n!}{(n+1)n!} && \text{(Express } n! \text{ as a factor of } (n+1)!.) \\ &= |x| \lim_{n\to\infty} \frac{1}{n+1} && \text{(Cancel common factors.)} \\ &= 0. \end{aligned}$$

We conclude that regardless of x, the power series in question converges absolutely. Consequently, the radius of convergence is $\rho = \infty$, and the interval of convergence is $I = (-\infty, \infty)$. ⋄

Example 2.7.5. Construct the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n}$.

Solution. We proceed by the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2(n+1)}{2n} \right| = \frac{x^2}{2} \lim_{n\to\infty} \frac{n+1}{n} = \frac{x^2}{2}$$

By the Ratio Test, the series converges if $L < 1$ if and only if $x^2 < 2$ if and only if $-\sqrt{2} < x < \sqrt{2}$. Even more, the series diverges if $L > 1$ if and only if $x^2 > 2$ if and only if $x > \sqrt{2}$ or $x < -\sqrt{2}$. Consequently, it suffices to determine convergence at $x = \pm\sqrt{2}$. Observe that if $x = \sqrt{2}$, then

$$\frac{x^{2n+1}}{2^n} = \frac{\sqrt{2}^{2n+1}}{\sqrt{2}^{2n}} = \sqrt{2} \text{ so that } \frac{(-1)^n x^{2n+1}}{2^n n} = \frac{\sqrt{2}}{n}.$$

Considering that this sequence is positive, decreasing, and converges to 0, by the Alternating Series Test, we conclude that the power series converges at $x = \sqrt{2}$. Likewise, if $x = -\sqrt{2}$, then

$$(-1)^n x^{2n+1} = (-1)^n(-\sqrt{2})^{2n+1} = (-1)^n(-1)^{2n+1}\sqrt{2}^{2n+1} = (-1)^{3n+1}\sqrt{2}^{2n+1} = (-1)^{n+1}\sqrt{2}^{2n+1}$$

is alternating. By the above rationale, the series converges at $x = -\sqrt{2}$. We conclude that the radius of convergence of the power series is $\rho = 2\sqrt{2}$, and the interval of convergence is $[-\sqrt{2}, \sqrt{2}]$. ⋄

Example 2.7.6. Construct the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} n^n x^n$.

Solution. We proceed by the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}x^{n+1}}{n^n x^n} \right| = |x| \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{n^n} = |x| \lim_{n\to\infty} (n+1)\left(\frac{n+1}{n}\right)^n$$

Consequently, it suffices to compute the limit of the terms involving n. We note first that if

$$y = \left(1+\frac{1}{x}\right)^x, \text{ then}$$

$$\ln(y) = x\ln\left(1+\frac{1}{x}\right) \text{ implies that}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right) + \ln\left(1+\frac{1}{x}\right) = -\frac{1}{x+1} + \ln\left(1+\frac{1}{x}\right) \text{ and}$$

$$\frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x \left[-\frac{1}{x+1} + \ln\left(1+\frac{1}{x}\right)\right] > 0 \text{ for all real numbers } x \geq 1.$$

Crucially, it follows that the sequence of n in the above limit is increasing and unbounded because it is a product of increasing sequences neither of which converges to 0. Consequently, we have that $L = \infty$. By the Ratio Test, we conclude that the power series diverges for all real numbers $x \neq 0$. ◇

Example 2.7.7. Construct the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(3-x)^n}{3^n}$.

Solution. By grouping each of the terms of the power series with exponent n together under the same exponent, we can view this power series as a geometric series with common ratio

$$r = \frac{x-3}{-3}.$$

By the Convergence of Geometric Series, we conclude that the power series converges if and only if $-1 < r < 1$ if and only if $-3 < x - 3 < 3$ if and only if $0 < x < 6$. Consequently, the radius of convergence of the power series is $\rho = 3$, and the interval of convergence is $I = (0, 6)$. ◇

Example 2.7.8. Given any nonzero real number c and any real function $g(x)$, consider the series

$$f(x) = \sum_{n=0}^{\infty} c[g(x)]^n.$$

By the Ratio Test, we have that $f(x)$ converges absolutely if and only if

$$|g(x)| = \lim_{n\to\infty} |g(x)| = \lim_{n\to\infty} \left| \frac{[g(x)]^{n+1}}{[g(x)]^n} \right| = \lim_{n\to\infty} \left| \frac{c[g(x)]^{n+1}}{c[g(x)]^n} \right| < 1.$$

Even more, by the Convergence of Geometric Series formula, if $|g(x)| < 1$, then

$$f(x) = \sum_{n=0}^{\infty} c[g(x)]^n = \frac{c}{1 - g(x)}.$$

Consequently, we obtain a closed form of $f(x)$ in terms of $g(x)$ for all real numbers x with $|g(x)| < 1$.

Example 2.7.9. Use Example 2.7.8 to express each of the following functions as a power series; then, state the radius and interval of convergence for each power series.

(a.) $\dfrac{1}{1-x}$ (b.) $\dfrac{1}{1+x}$ (c.) $\dfrac{1}{1+x^2}$

Solution. (a.) Observe that for the real function $g(x) = x$, we obtain the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

that is valid for all real numbers x such that $|x| < 1$. We conclude that the radius of convergence of the power series is $\rho = 1$, and the interval of convergence is $I = (-1, 1)$.

(c.) Observe that for the real function $g(x) = -x^2$, we obtain the power series representation

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

that is valid for all real numbers x such that $0 \leq x^2 < 1$. Considering that $0 \leq x^2 < 1$ if and only if $-1 < x < 1$, the radius of convergence is $\rho = 1$, and the interval of convergence is $I = (-1, 1)$. ⋄

One of the most useful features of power series is that we may differentiate them term-by-term.

Theorem 2.7.10 (Power Series Are Differentiable)**.** *Consider the following power series centered at a real number c with (possibly infinite) radius of convergence $\rho > 0$.*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

(a.) *We have that $f(x)$ is differentiable on the interval $I = (c - \rho, c + \rho)$ with derivative*

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dx} (x-c)^n = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}.$$

Consequently, $f'(x)$ is a power series centered at $x = c$ with radius of convergence ρ.

(b.) *We have that the antiderivative of $f(x)$ on the interval $I = (c - \rho, c + \rho)$ is given by*

$$F(x) + C = \int f(x)\,dx = \int \sum_{n=0}^{\infty} a_n (x-c)^n\,dx = \sum_{n=0}^{\infty} a_n \int (x-c)^n\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}.$$

Consequently, $\int f(x)\,dx$ is a power series centered at $x = c$ with radius of convergence ρ.

We note that in practice, the constant C can be found by used the fact that $F(c) + C = 0$.

Example 2.7.11. Use Example 2.7.9 to find a power series representation for the following functions; then, state the radius and interval of convergence for each power series.

(a.) $\dfrac{1}{(1-x)^2}$ (b.) $\dfrac{-2x}{(1+x^2)^2}$ (c.) $\ln(1+x)$ (d.) $\arctan(x)$

Solution. (a.) By the Power Rule and the Chain Rule for Derivatives, we have that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$$

by Theorem 2.7.10; the radius of convergence is $\rho = 1$, and the interval of convergence is $(-1, 1)$.

(b.) By the Power Rule and Chain Rule for Derivatives, we have that

$$\frac{-2x}{(1+x^2)^2} = \frac{d}{dx}\frac{1}{1+x^2} = \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n \frac{d}{dx} x^{2n} = \sum_{n=1}^{\infty}(-1)^n 2nx^{2n-1}$$

with radius of convergence $\rho = 1$ and interval of convergence $(-1, 1)$.

(c.) Observe that if $-1 < x < 1$, then $0 < 1 + x < 2$ and

$$\ln(1+x) + C = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^n\,dx = \sum_{n=0}^{\infty}(-1)^n \int x^n\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

By plugging in $x = 0$, we find that $C = \ln(1) + C = 0$, hence we conclude that

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

with radius of convergence $\rho = 1$ and interval of convergence $[-1, 1]$. Crucially, we achieve convergence at both endpoints $x = -1$ and $x = 1$ by the Alternating Series Test.

(d.) Last, we have that

$$\arctan(x) + C = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

By plugging in $x = 0$, we find that $C = \arctan(0) + C = 0$, hence we conclude that

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

with radius $\rho = 1$ and interval of convergence $[-1, 1]$ by the Alternating Series Test. ◇

One immediate consequence of the previous example is that we may approximate (via power series) the value of previously unknown quantities. By Example 2.7.11(c.), we have that

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

Consequently, we can approximate π to any desired degree of accuracy, i.e., we can write the decimal expansion of π in a manner that is accurate to as many decimal places as desired! We remark that the convergence of the power series expansion of $\arctan(1)$ is very slow: indeed, it is asymptotically equivalent to the alternating harmonic series, so one would require a better series approximation of π in practice. Even still, this expansion of π is historically significant and quite remarkable.

Consider any real function $f(x)$ and any real number c such that the nth derivative $f^{(n)}(x)$ of $f(x)$ exists at $x = c$ for all integers $n \geq 0$. We will henceforth refer to the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

as the **Taylor series** of $f(x)$ centered at $x = c$. Crucially, if a real function $f(x)$ is represented by a power series centered at $x = c$ on some open interval of the form $(c - \rho, c + \rho)$ for some real number $\rho > 0$, then that power series must be the Taylor series of $f(x)$ centered at $x = c$.

Theorem 2.7.12 (Uniqueness of Taylor Series)**.** *Given any real function $f(x)$ for which the Taylor series of $f(x)$ centered at $x = c$ exists, there exists a real number $\rho > 0$ such that the Taylor series of $f(x)$ centered at $x = c$ is the unique power series expansion of $f(x)$ on the interval $(c - \rho, c + \rho)$.*

Caution: be very careful not to misinterpret the theorem. Explicitly, this does not guarantee that a real function $f(x)$ admits a power series expansion; rather, it says that if $f(x)$ can be represented as a power series center at $x = c$, then that power series is in fact the Taylor series of $f(x)$.

We have in Examples 2.7.8, 2.7.9, and 2.7.11 provided power series expansions of the logarithmic function $\ln(1 + x)$, the inverse trigonometric function $\arctan(x)$, and the rational functions

$$\frac{1}{1-x} \text{ and } \frac{1}{1+x^2}.$$

Consequently, by Theorem 2.7.12, these are in fact the Taylor series expansions of these functions centered at $x = 0$! Generally, if it exists, the Taylor series expansion of a real function $f(x)$ centered at $x = 0$ is referred to as the **Maclaurin series** of $f(x)$; the terminology is no doubt perplexing, but it is commonplace and remains in use due to historical considerations.

Example 2.7.13. Construct the Maclaurin series of $f(x) = e^x$.

Solution. We compute the nth derivative of e^x for each integer $n \geq 0$. Considering that $f(x) = e^x$ satisfies that $f'(x) = e^x$, we find that $f^{(n)}(x) = e^x$ for all integers $n \geq 0$. Consequently, we have that $f^{(n)}(0) = 1$ for all integers $n \geq 0$. By the formula for the Maclaurin series, we conclude that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Even more, by the Ratio Test, the series converges absolutely for all real numbers since

$$L = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \left| \frac{xn!}{(n+1)n!} \right| = |x| \lim_{n \to \infty} \frac{1}{n+1} = 0. \qquad \diamond$$

Example 2.7.14. Construct the Maclaurin series of $f(x) = \cos(x)$.

Solution. We compute the nth derivative of $\cos(x)$ for each integer $n \geq 0$. Observe that

$$f(x) = \cos(x), \qquad f''(x) = -\cos(x), \text{ and}$$
$$f'(x) = -\sin(x), \qquad f'''(x) = \sin(x).$$

Considering that $\cos(x)$ is the derivative of $\sin(x)$, the derivatives of $\cos(x)$ are periodic with

$$f^{(4k)}(x) = \cos(x), \qquad f^{(4k+2)}(x) = -\cos(x), \text{ and}$$
$$f^{(4k+1)}(x) = -\sin(x), \qquad f^{(4k+3)}(x) = \sin(x)$$

for all integers $k \geq 0$. Consequently, for each integer $k \geq 1$, we have that

$$f^{(4k)}(0) = 1, \qquad f^{(4k+2)}(0) = -1, \text{ and}$$
$$f^{(4k+1)}(0) = 0, \qquad f^{(4k+3)}(0) = 0.$$

Put another way, the even derivatives of $f(x) = \cos(x)$ are alternating in sign and the odd derivatives of $f(x) = \cos(x)$ are zero for $x = 0$. By the formula for the Maclaurin series, we conclude that

$$\cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Even more, by the Ratio Test, the series converges absolutely for all real numbers since

$$L = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2 (2n)!}{(2n+2)(2n+1)(2n)!} \right| = x^2 \lim_{n\to\infty} \frac{1}{4n^2+6n+2} = 0. \qquad \diamond$$

Example 2.7.15. Construct the Maclaurin series of $f(x) = \sin(x)$.

Solution. We compute the nth derivative of $\sin(x)$ for each integer $n \geq 0$. Observe that

$$f(x) = \sin(x), \qquad f''(x) = -\sin(x), \text{ and}$$
$$f'(x) = \cos(x), \qquad f'''(x) = -\cos(x).$$

Considering that $\sin(x)$ is the derivative of $-\cos(x)$, the derivatives of $\sin(x)$ are periodic with

$$f^{(4k)}(x) = \sin(x), \qquad f^{(4k+2)}(x) = --\sin(x), \text{ and}$$
$$f^{(4k+1)}(x) = \cos(x), \qquad f^{(4k+3)}(x) = -\cos(x)$$

for all integers $k \geq 0$. Consequently, for each integer $k \geq 1$, we have that

$$f^{(4k)}(0) = 0, \qquad f^{(4k+2)}(0) = 0, \text{ and}$$
$$f^{(4k+1)}(0) = 1, \qquad f^{(4k+3)}(0) = -1.$$

Put another way, the odd derivatives of $f(x) = \sin(x)$ are alternating in sign and the even derivatives of $f(x) = \sin(x)$ are zero for $x = 0$. By the formula for the Maclaurin series, we conclude that

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Even more, by the Ratio Test, the series converges absolutely for all real numbers since

$$L = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2 (2n+1)!}{(2n+3)(2n+2)(2n+1)!} \right| = x^2 \lim_{n\to\infty} \frac{1}{4n^2+10n+6} = 0. \ \diamond$$

Theorem 2.7.16 (Convergence of Taylor Series)**.** *Consider any real numbers c and $\rho > 0$ and any real function $f(x)$ such that $f^{(n)}(x)$ is continuously differentiable for all integers $n \geq 0$ and all real numbers x such that $c - \rho < x < c + \rho$. Provided that there exists a real number K such that $|f^{(n)}(x)| \leq K$ for all integers $n \geq 0$ and all real numbers x such that $c - \rho < x < c + \rho$, the Taylor series of $f(x)$ centered at $x = c$ converges to $f(x)$, i.e., the following representation of $f(x)$ is valid.*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-c)^n$$

Once we know the Taylor series expansion of some real function $f(x)$ centered at $x = c$, it is not overtly difficult to find the Taylor series expansion of the real functions $x^k f(x)$ or $f(x)/x^k$ for some any integer $k \geq 1$ or the Taylor series expansion of $(g \circ f)(x)$ for some real function $g(x)$; however, it is possible to change the center of a Taylor series when performing these operations.

Example 2.7.17. Construct the Taylor series expansion of each of the following.

(a.) $x^3 \cos(x)$ (b.) e^{1-x^2} (c.) e^{x-4} (d.) $\dfrac{x - \sin(x)}{x}$

Solution. (a.) By Example 2.7.14, the Maclaurin series for $\cos(x)$ is given by

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

By multiplying this power series by x^3, we obtain the Maclaurin series expansion

$$x^3 \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n)!}.$$

By the Ratio Test, we conclude that this series converges for all real numbers.

(b.) By Example 2.7.13, the Maclaurin series for e^x is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Considering that $e^{1-x^2} = ee^{-x^2}$, plugging in $-x^2$ to the above yields the Maclaurin series

$$e^{1-x^2} = ee^{-x^2} = e \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e(-1)^n x^{2n}}{n!}.$$

By the Ratio Test, we conclude that this series converges for all real numbers.

(c.) Likewise, by plugging in $x - 4$ to the Maclaurin series of e^x, we obtain the Taylor series

$$e^{x-4} = \sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$$

of e^x centered at $x = 4$. By the Ratio Test, this series converges for all real numbers.

(d.) By Example 2.7.15, the Maclaurin series for $\sin(x)$ is given as follows.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

By multiplying this expansion by -1 and adding x, we obtain the following.

$$x - \sin(x) = x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots$$

Last, by dividing each side of this identity by x, we conclude that

$$\frac{x - \sin(x)}{x} = \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n+1)!}.$$

By the Ratio Test, we conclude that this Maclaurin series converges for all real numbers. ◇

2.8 Power Series Solutions of Linear Equations

We return our attention to the following homogeneous second order linear equation.

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{2.8.1}$$

We will refer to a real number x_0 in the domain of $P(x)$ as an **ordinary point** of Equation (2.8.1) provided that $P(x_0)$ is nonzero. Conversely, every point x_0 in the domain of $P(x)$ for which $P(x_0)$ is zero is called a **singular point** of Equation (2.8.1). We note that by definition, if x_0 is an ordinary point of Equation (2.8.1), then the functions $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are defined in a neighborhood of x_0 so long as $P(x)$ is continuous in a neighborhood of x_0. We will assume as usual that each of the functions $P(x)$, $Q(x)$, and $R(x)$ is continuous in an open interval containing the real number x_0, hence we may reduce to the familiar homogeneous second order linear equation

$$y'' + p(x)y' + q(x)y = 0. \tag{2.8.2}$$

We have until this point in the course rarely considered the situation in which $p(x)$ and $q(x)$ are non-constant functions; however, we will develop in this section a new technique that allows us to relax these assumptions a bit to encompass the case that $p(x)$ and $q(x)$ are polynomial functions. Crucially, we assume that a solution $y = \phi(x)$ of Equation (2.8.2) admits a Taylor series

$$\phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{2.8.3}$$

that is defined on an open interval. Considering that Power Series Are Differentiable, we have that

$$\phi'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \text{ and} \tag{2.8.4}$$

$$\phi''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}. \tag{2.8.5}$$

Consequently, by substituting $y = \phi(x)$ and its derivatives into Equation (2.8.2), we find that

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + p(x)\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} + q(x)\sum_{n=0}^{\infty} a_n(x-x_0)^n. \tag{2.8.6}$$

Equation (2.8.6) is especially tractable in the case that $p(x)$ and $q(x)$ are polynomial functions. We illustrate the mechanism first in the following example before laying out the general approach.

Example 2.8.1. Construct the general solution of the following second order linear equation.

$$y'' - y = 0$$

Solution. Certainly, we can approach this equation as we have done since Section 2.1: the characteristic equation $(r-1)(r+1) = r^2 - 1 = 0$ yields the general solution of $y = \gamma(x) = c_1 e^x + c_2 e^{-x}$. Bearing this in mind, it follows that some solution of the equation admits a power series expansion (since this is a sum of functions that admit Taylor series). Consequently, we will assume that

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series solution of the equation at hand. Considering that $\phi''(x) = \phi(x)$, it follows that

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n.$$

Consider the change of index $k = n-2$. We find that $k = 0$ whenever $n = 2$ and $n = k+2$ so that

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k = \sum_{n=0}^{\infty} a_n x^n.$$

By viewing the indices k and n as dummy variables, the coefficient $(n+2)(n+1)a_{n+2}$ of the monomial x^n on the left-hand side and the coefficient a_n of the monomial x^n on the right-hand side must be equal for each integer $n \geq 0$ by the Uniqueness of Taylor Series Theorem. We conclude that

$$(n+2)(n+1)a_{n+2} = a_n \text{ or } a_{n+2} = \frac{a_n}{(n+2)(n+1)}. \tag{2.8.7}$$

Equation (2.8.7) is distinguished as a **recurrence relation**; its significance lies in the fact that we may repeatedly invoke the relation to write a_{n+2} in terms of the preceding elements of the sequence.

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \qquad a_n = \frac{a_{n-2}}{n(n-1)} \qquad a_{n-2} = \frac{a_{n-4}}{(n-2)(n-1)}$$

Continuing this pattern of reduction, we must eventually come to the terms a_0 or a_1 of the sequence. Crucially, this distinction depends on the **parity** (i.e., evenness or oddness) of the positive integer $n \geq 2$. Explicitly, we will assume that $k \geq 0$ is an integer for which $n = 2k$ or $n = 2k+1$.

$$a_{2k} = \frac{a_{2k-2}}{(2k)(2k-1)} = \frac{a_{2k-4}}{(2k)(2k-1)(2k-2)(2k-3)} = \cdots = \frac{a_0}{(2k)!}$$

$$a_{2k+1} = \frac{a_{2k-1}}{(2k+1)(2k)} = \frac{a_{2k-3}}{(2k+1)(2k)(2k-1)(2k-2)} = \cdots = \frac{a_1}{(2k+1)!}$$

Consequently, we find that the series expansion of $\phi(x)$ is given by

$$\phi(x) = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{(2k+1)}}{(2k+1)!}.$$

Each of the power series that determine $\phi(x)$ converges for all real numbers by the Ratio Test, hence they can differentiated term-by-term since Power Series Are Differentiable; however, it is not too difficult to see that $\phi(x) = c_1 e^x + c_2 e^{-x}$ for some real numbers c_1 and c_2. Explicitly, we have that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Considering that function composition is a valid power series operation, we find that

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}.$$

Consequently, the even terms of e^{-x} are positive and the odd terms are negative so that

$$\frac{1}{2}(e^x + e^{-x}) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

By the same rationale, we may determine a function representation of the other power series.

$$\frac{1}{2}(e^x - e^{-x}) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

We conclude at last that the function and power series representations are compatible.

$$\phi(x) = \frac{a_0}{2}(e^x + e^{-x}) + \frac{a_1}{2}(e^x - e^{-x}) = \frac{a_0 + a_1}{2} e^x + \frac{a_0 - a_1}{2} e^{-x} \qquad \diamond$$

Algorithm 2.8.2 (Power Series Solutions Algorithm)**.** Consider any second order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

defined for some real univariate functions $p(x)$ and $q(x)$ that are continuous in a neighborhood of x_0. Carry out the following steps to derive a power series solution of the differential equation.

1.) Consider some solution $y = \phi(x)$ of the equation that admits a power series representation.

$$\phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

2.) Compute the first and second derivatives of $\phi(x)$ using term-by-term differentiation.

$$\phi'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$\phi''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

3.) Construct the power series equation $\phi'' + p(x)\phi' + q(x)\phi = 0$.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + p(x)\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} + q(x)\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$$

4.) Carry out a change of indices to express the above as one sum beginning at an integer $n_0 \geq 0$.

5.) Completely solve the recurrence relation induced by the above power series equation.

6.) Construct the radius and interval of convergence of the power series that define $\phi(x)$.

7.) Check that the power series that define $\phi(x)$ form a fundamental set of solutions for Equation (2.8.2) by verifying that their Wronskian is nonzero at some point in their domain.

Example 2.8.3 (Airy's Equation)**.** Construct the general solution of the following equation.

$$y'' - xy = 0$$

Solution. Before we begin our solution, we remark on the novelty of this example: we have yet to attempt a second order equation with non-constant coefficients; this marks our very first! We will proceed according to the Power Series Solutions Algorithm, hence we must first assume that

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series solution of Airy's Equation. Consequently, the identity $\phi''(x) = x\phi(x)$ yields that

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \tag{2.8.8}$$

Crucially, we cannot directly compare the coefficients of the monomials x^n and x^{n+1} because they have different degrees; however, using the change of index $k = n+1$, we find that

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=1}^{\infty} a_{k-1}x^k.$$

By viewing k as a dummy variable in the second power series and readopting the index $n \geq 1$, we place ourselves in a situation to compare the two power series from the above equation (2.8.8).

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n$$

Considering the Uniqueness of Taylor Series Theorem, for each integer $n \geq 1$, we must have that

$$2a_2 = 0 \text{ and} \tag{2.8.9}$$

$$(n+2)(n+1)a_{n+2} = a_{n-1}. \tag{2.8.10}$$

We note that a_{n+2} is the third term of the sequence after a_{n-1}, hence it follows that the sequence a_n is determined by a_{3k}, a_{3k+1}, and a_{3k+2} for each integer $k \geq 0$. Explicitly, the following hold.

$$a_{3k} = \frac{a_{3k-3}}{(3k)(3k-1)} = \frac{a_{3k-6}}{(3k)(3k-1)(3k-3)(3k-4)} = \cdots = \frac{a_0}{(3k)(3k-1)\cdots 6 \cdot 5 \cdot 3 \cdot 2}$$

$$a_{3k+1} = \frac{a_{3k-2}}{(3k+1)(3k)} = \frac{a_{3k-5}}{(3k+1)(3k)(3k-2)(3k-3)} = \cdots = \frac{a_1}{(3k+1)(3k)\cdots 7 \cdot 6 \cdot 4 \cdot 3}$$

$$a_{3k+2} = \frac{a_{3k-1}}{(3k+2)(3k+1)} = \frac{a_{3k-4}}{(3k+2)(3k+1)(3k-1)(3k)} = \cdots = \frac{a_2}{(3k+2)(3k+1)\cdots 8 \cdot 7 \cdot 5 \cdot 4}$$

Considering that $a_2 = 0$, it follows that $a_{3k+2} = 0$ for each integer $k \geq 0$, hence we conclude that

$$\phi(x) = a_0 \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)(3k-1)\cdots 6 \cdot 5 \cdot 3 \cdot 2} + a_1 \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)(3k)\cdots 7 \cdot 6 \cdot 4 \cdot 3}$$

for some real numbers a_0 and a_1. Even more, by the Ratio Test, it can readily be established that each of the power series the define $\phi(x)$ converges for all real numbers. Computing the derivatives of the attendant power series by differentiating term-by-term yields the following.

$$\frac{d}{dx}\left[a_0 \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)(3k-1)\cdots 6 \cdot 5 \cdot 3 \cdot 2}\right] = a_0 \sum_{k=1}^{\infty} \frac{x^{3k-1}}{(3k-1)(3k-3)(3k-4)\cdots 6 \cdot 5 \cdot 3 \cdot 2}$$

$$\frac{d}{dx}\left[a_1 \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)(3k)\cdots 7 \cdot 6 \cdot 4 \cdot 3}\right] = a_1 \sum_{k=1}^{\infty} \frac{x^{3k}}{(3k)(3k-2)(3k-3)\cdots 7 \cdot 6 \cdot 4 \cdot 3}$$

Consequently, if we denote the power series with coefficient a_0 as $\phi_0(x)$ and the power series with coefficient a_1 as $\phi_1(x)$, then $\phi(x) = a_0\phi_0(x) + a_1\phi_1(x)$ with $\phi_0(0) = 1$, $\phi_0'(0) = 0$, $\phi_1(0) = 0$, and $\phi_1'(0) = \frac{1}{3}$. Crucially, it follows that $W(\phi_0, \phi_1)(0) = \phi_0(0)\phi_1'(0) - \phi_0'(0)\phi_1(0)$ is nonzero. Choosing $a_0 = 1$ and $a_1 = 0$ yields that $\phi_0(x)$ is a solution of Airy's Equation; likewise, the coefficients $a_0 = 0$ and $a_1 = 1$ establish that $\phi_1(x)$ is a solution of Airy's Equation. We may therefore appeal to Abel's Theorem to deduce that $\phi_0(x)$ and $\phi_1(x)$ form a fundamental set of solutions of Airy's Equation. ◇

Caution: Example 2.8.3 lays out the most elementary solution to Airy's Equation that is currently known; in fact, the functions $\phi_0(x)$ and $\phi_1(x)$ are typically referred to as Airy's functions!

Example 2.8.4. Construct the general solution of the following equation.

$$xy'' + y' + xy = 0$$

Solution. Considering that $P(x) = x$, it follows that $x = 0$ is a singular point of the above equation. We may therefore assume that $y = \phi(x)$ is a power series solution centered at $x_0 = 1$

$$\phi(x) = \sum_{n=0}^{\infty} a_n(x-1)^n.$$

Consequently, the identity $x\phi''(x) + \phi'(x) + x\phi(x) = 0$ yields that

$$x \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + x \sum_{n=0}^{\infty} a_n(x-1)^n = 0.$$

Before we obtain a recurrence relation on the coefficients a_n, we must simplify the above equation. Crucially, the center of each power series at hand is $x_0 = 1$ (i.e., these are not Maclaurin series), hence we must use the fact that $x = (x-1)+1$ to negotiate the terms with a coefficient of x.

$$x \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = (x-1) \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

$$x \sum_{n=0}^{\infty} a_n(x-1)^n = (x-1) \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^n$$

$$= \sum_{n=0}^{\infty} a_n(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n(x-1)^n$$

We seek next to ensure that all powers of the monomial x^n are equal; we achieve this by either a change of index (when appropriate) or by evaluating the first few terms of each series.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} = \sum_{n=1}^{\infty} (n+1)na_{n+1}(x-1)^n \qquad \text{(change of index } k = n-1\text{)}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n \qquad \text{(change of index } k = n-2\text{)}$$

$$= 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n \qquad \text{(Plug in } n = 0.\text{)}$$

$$\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n \qquad \text{(change of index } k = n-1\text{)}$$

$$= a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1}(x-1)^n \qquad \text{(Plug in } n = 0.\text{)}$$

$$\sum_{n=0}^{\infty} a_n(x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1}(x-1)^n \qquad \text{(change of index } k = n+1)$$

$$\sum_{n=0}^{\infty} a_n(x-1)^n = a_0 + \sum_{n=1}^{\infty} a_n(x-1)^n \qquad \text{(Plug in } n = 0.)$$

Collectively, the sum of each of the infinite series beginning with $n = 1$ is zero, according to the original differential equation $xy'' + y' + xy = 0$ and our simplification, hence we find that

$$2a_2 + a_1 + a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + a_n + a_{n-1}](x-1)^n = 0.$$

Considering that the power series expansion of a function is unique (if it exists) by the Uniqueness of Taylor Series Theorem, we conclude that for each integer $n \geq 1$, we must have that

$$2a_2 + a_1 + a_0 = 0 \text{ and} \qquad (2.8.11)$$

$$(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + a_n + a_{n-1} = 0. \qquad (2.8.12)$$

Equation (2.8.11) yields that $a_2 = -\frac{1}{2}(a_0 + a_1)$ and Equation (2.8.12) simplifies to

$$a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+2)(n+1)}. \qquad (2.8.13)$$

Unfortunately, Equation (2.8.13) represents a fourth order non-linear recurrence relation, so there is little hope for solving a_{n+2} in terms of a_0, a_1, and a_2; however, it is possible in this case to determine the first four terms of the power series solutions $\phi_0(x)$ and $\phi_1(x)$ that form a fundamental set of solutions of the equation at hand. Explicitly, if we set $a_0 = 1$ and $a_1 = 0$, then $a_2 = -\frac{1}{2}$ and

$$a_3 = -\frac{4a_2 + a_1 + a_0}{6} = -\frac{-2+0+1}{6} = \frac{1}{6}$$

since this term is a_{n+2} with $n = 1$. Likewise, if we set $a_0 = 0$ and $a_1 = 1$, then $a_2 = -\frac{1}{2}$ and

$$a_3 = -\frac{4a_2 + a_1 + a_0}{6} = -\frac{-2+1+0}{6} = \frac{1}{6}.$$

We conclude that $\phi_0(x)$ and $\phi_1(x)$ are determined by their first four terms as given by

$$\phi_0(x) = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \cdots \text{ and } \phi_1(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \cdots.$$

One can check that $\phi_0(1) = 1$, $\phi_0'(1) = 0$, $\phi_1(1) = 0$, and $\phi_1'(1) = 1$, hence $\phi_0(x)$ and $\phi_1(x)$ form a fundamental set of solutions for the given differential equation by Abel's Theorem. ⋄

Caution: Example 2.8.4 illustrates that it is not always possible to solve the recurrence relation obtained by the Power Series Solutions Algorithm; however, we can always determine the value of the first n terms $a_0, a_1, \ldots, a_{n-1}$ for as large a value of $n \geq 1$ as desired.

2.9 Existence of Power Series Solutions of Linear Equations

We turn our attention in this section to the question of existence of power series solutions of second order linear differential equations. Consider the second order linear ordinary differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{2.9.1}$$

defined for some real univariate functions $P(x)$, $Q(x)$, and $R(x)$ that are continuous in a neighborhood of some real number x_0 for which $P(x_0)$ is nonzero. We refer to x_0 as an ordinary point of the differential equation (2.9.1). We assume that $y = \phi(x)$ is a power series solution of Equation (2.9.1) centered at $x = x_0$ and defined on some open interval $(x_0 - \rho, x_0 + \rho)$ for some (possibly infinite) real number $\rho > 0$. Consequently, there exists a real sequence a_n such that

$$\phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges for each real number x such that $x_0 - \rho < x < x_0 + \rho$. Crucially, this entails that

$$\begin{aligned} \phi(x_0) &= a_0, & \phi''(x_0) &= 2a_2, \\ \phi'(x_0) &= a_1, & \phi'''(x_0) &= 6a_3, \end{aligned}$$

and $\phi^{(k)}(x_0) = k! a_k$ for each integer $k \geq 0$. On the other hand, if $\phi(x)$ is a solution of Equation (2.9.1), then we must have that $P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0$ in a neighborhood of x_0. Considering that $P(x)$ is continuous in a neighborhood of x_0 and $P(x_0)$ is nonzero, it follows that

$$\phi''(x) + p(x)\phi'(x) + q(x)\phi(x) = 0$$

in a neighborhood of x_0 for the real univariate functions $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. We may therefore determine the derivatives of $\phi(x)$ at x_0 via the Product Rule so long as we know the values of $\phi(x_0)$ and $\phi'(x_0)$ since the above equation can be rewritten as

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x).$$

Explicitly, we may compare the real numbers $k! a_k$ and $\phi^{(k)}(x_0)$ for each integer $k \geq 2$ according to

$$k! a_k = \phi^{(k)}(x_0) = \frac{d^{k-2}}{dx^{k-2}} \phi''(x) = \frac{d^{k-2}}{dx^{k-2}} [-p(x)\phi'(x) - q(x)\phi(x)]_{x = x_0}.$$

Ostensibly, in order to ensure that the above equation is well-defined, it seems sufficient to assume that $p(x)$ and $q(x)$ are infinitely differentiable at x_0; however, this is not enough to guarantee that the coefficients a_k determine a convergent power series. We require the stronger condition that $p(x)$ and $q(x)$ are **analytic** at x_0, i.e., the real univariate functions $p(x)$ and $q(x)$ admit Taylor series expansions centered at $x = x_0$. We have thus far in the course only encountered analytic functions, so this is certainly not asking for too much, but it rules out a vast majority of real functions.

Theorem 2.9.1 (Fundamental Theorem of Power Series Solutions of Homogeneous Second Order Linear Ordinary Differential Equations)**.** *Consider the second order linear differential equation*

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

defined for some real univariate functions $P(x)$, $Q(x)$, *and* $R(x)$ *that are continuous in a neighborhood of some real number* x_0 *such that* $P(x_0)$ *is nonzero. If the quotient functions* $p(x) = Q(x)/P(x)$ *and* $q(x) = R(x)/P(x)$ *are analytic at* x_0, *then the general solution* $y = \phi(x)$ *of the above differential equation admits a power series expansion centered at* $x = x_0$, *i.e., we have that*

$$\phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Even more, there exist analytic functions $\phi_0(x)$ *and* $\phi_1(x)$ *that form a fundamental set of solutions of the above differential equation in a neighborhood of* x_0, *i.e., we have that*

$$\phi(x) = a_0 \phi_0(x) + a_1 \phi_1(x).$$

Last, if the radii of convergence of the analytic functions $p(x)$ *and* $q(x)$ *are some real numbers* ρ_p *and* ρ_q, *respectively, then the radii of convergence of* $\phi_0(x)$ *and* $\phi_1(x)$ *are at least* $\min\{\rho_p, \rho_q\}$.

We will not endeavor to prove this result (indeed, fundamental theorems are often quite difficult to prove), but we point out that this theorem is due to the German mathematician Lazarus Fuchs in the nineteenth century. Using the Fundamental Theorem of Power Series Solutions of Homogeneous Second Order Linear Ordinary Differential Equations as a black box, we are now in a position to say that as long as the coefficients of Equation (2.9.1) are analytic at an ordinary point in the domain of $P(x)$, then there is a power series solution of the equation whose radius of convergence is at least as small as the minimum of the radii of convergence of the quotient functions $Q(x)/P(x)$ and $R(x)/P(x)$. We illustrate this technique in the following examples; the reader is encouraged to notice the method of passing to the complex plane to determine the radius of convergence (indeed, the radius of convergence is the radius of the domain of the power series in the complex plane).

Example 2.9.2. Construct a lower bound for the radius of convergence of the fundamental set of solutions of the following homogeneous second order linear differential equation.

$$y'' + xy' + x^2 y = 0$$

Solution. We note that $P(x) = 1$, $Q(x) = x$, and $R(x) = x^2$, hence the quotient functions that define the given differential equation are $p(x) = x$ and $q(x) = x^2$; these are polynomial functions, so they are analytic for all real numbers so that $\rho_p = \rho_q = \infty$. By the Fundamental Theorem of Power Series Solutions of Homogeneous Second Order Linear Ordinary Differential Equations, the radius of convergence of the fundamental set of solutions of the above equation is $\rho = \infty$. ⋄

Example 2.9.3. Construct a lower bound for the radius of convergence of the fundamental set of solutions of the following homogeneous second order linear differential equation.

$$(1 - x)^2 y'' + xy' - y = 0$$

Solution. We note that $P(x) = (1 - x)^2$, $Q(x) = x$, and $R(x) = -1$, hence the quotient functions that define the given differential equation are $p(x) = x/(1-x)^2$ and $q(x) = -1/(1-x)^2$. We provide two different approaches to determine the radii of convergence of $p(x)$ and $q(x)$. On the one hand, we may view $p(x)$ as a product of the derivative of the usual geometric series $1/(1-x)$ with x.

$$p(x) = \frac{x}{(1-x)^2} = x \cdot \frac{d}{dx}\frac{1}{1-x} = x \cdot \frac{d}{dx}\sum_{n=0}^{\infty} x^n = x\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n$$

Consequently, by the Ratio Test, the power series expansion of $p(x)$ converges for all real numbers $|x| < 1$ so that $\rho_p = 1$. Likewise, the above computation gives the power series expansion of $q(x)$.

$$q(x) = -\frac{1}{(1-x)^2} = -\frac{d}{dx}\frac{1}{1-x} = -\frac{d}{dx}\sum_{n=0}^{\infty} x^n = -\sum_{n=1}^{\infty} nx^{n-1}$$

Like before, the Ratio Test yields that $\rho_q = 1$. Considering $p(x)$ and $q(x)$ as complex functions, their domains in the complex plane include all complex numbers for which their denominators are nonzero. Consequently, the power series expansions of $p(x)$ and $q(x)$ converge at least for all real numbers such that $|x| < 1$ because the only real zero of $(1-x)^2$ is $x = 1$. Either way, we conclude that the radius of convergence of the fundamental set of solutions of the given equation is $\rho \geq 1$. ⋄

Example 2.9.4 (Legendre's Equation)**.** Construct a lower bound for the radius of convergence of the fundamental set of solutions of the following homogeneous second order linear differential equation (known as **Legendre's Equation**) for any real number α.

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

Solution. We note that $P(x) = 1 - x^2$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha+1)$, hence the quotient functions that define this differential equation are $p(x) = -2x/(1-x^2)$ and $q(x) = \alpha(\alpha+1)/(1-x^2)$. Like before, it is possible to determine the radii of convergence of $p(x)$ and $q(x)$ by expressing these rational functions as power series: both $p(x)$ and $q(x)$ are induced by the geometric series $1/(1-x)$.

$$p(x) = \frac{-2x}{1-x^2} = -2x \cdot \frac{1}{1-x^2} = -2x\sum_{n=0}^{\infty} x^{2n} = -2\sum_{n=0}^{\infty} x^{2n+1}$$

$$q(x) = \frac{\alpha(\alpha+1)}{1-x^2} = \alpha(\alpha+1) \cdot \frac{1}{1-x^2} = \alpha(\alpha+1)\sum_{n=0}^{\infty} x^{2n}$$

By the Ratio Test, each of the above power series converges for all real numbers such that $|x^2| < 1$, hence the interval of convergence of $p(x)$ and $q(x)$ is given by $(-1, 1)$ so that $\rho_p = \rho_q = 1$. Considering $p(x)$ and $q(x)$ as complex functions that are defined so long as $1-x^2$ is nonzero, we find that $x \neq \pm 1$. Consequently, the largest disk of convergence centered at the origin in the complex plane extends from $x = -1$ to $x = 1$ along the real axis, hence its radius is 1. Either way, we conclude that the radius of convergence of the fundamental set of solutions of the given equation is $\rho \geq 1$. ⋄

Example 2.9.5. Construct a lower bound for the radius of convergence of the fundamental set of solutions of the following homogeneous second order linear differential equation.

$$(1+x^3)y'' + 4xy' + y = 0$$

Solution. We note that $P(x) = 1 + x^3$, $Q(x) = 4x$, and $R(x) = 1$, hence the quotient functions that define the given differential equation are $p(x) = 4x/(1 + x^3)$ and $q(x) = 1/(1 + x^3)$. We have that

$$p(x) = \frac{4x}{1+x^3} = 4x \cdot \frac{1}{1-(-x^3)} = 4x \sum_{n=0}^{\infty} (-1)^n x^{3n} \text{ and}$$

$$q(x) = \frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-1)^n x^{3n}.$$

Each of these power series converges for all real numbers such that $|x^3| < 1$ by the Ratio Test, hence the radii of convergence for $p(x)$ and $q(x)$ are $\rho_p = \rho_q = 1$. By the Fundamental Theorem of Power Series Solutions of Homogeneous Second Order Linear Ordinary Differential Equations, the radius of convergence of the fundamental set of solutions of the given equation is $\rho \geq 1$.

Before we conclude this example and the section, let us assume next that we seek a fundamental set of solutions of the given equation for the ordinary point $x_0 = \frac{1}{2}$ instead of $x_0 = 0$. Going through the power series expansions of $p(x)$ and $q(x)$ centered at $x_0 = \frac{1}{2}$ is still possible here, but it is rather cumbersome: indeed, the simplest way would be to obtain $p(x)$ and $q(x)$ from some geometric series centered at $x_0 = \frac{1}{2}$, so we would have to recognize $1 + x^3$ in terms of some polynomial in $x - \frac{1}{2}$. On the other hand, if we view $x^3 + 1$ as a complex polynomial, then by the Quadratic Formula, its roots are $x = -1$ and $x = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$ since $x^3 + 1 = (x + 1)(x^2 - x + 1)$. Consequently, the distance from $x_0 = \frac{1}{2}$ to the nearest root of $x^3 + 1$ in the complex plane is $\frac{\sqrt{3}}{2}$, hence the fundamental set of solutions of the given equation centered at $x_0 = \frac{1}{2}$ converge for all real numbers $\left|x - \frac{1}{2}\right| < \frac{\sqrt{3}}{2}$. ◇

2.10 Euler Equations

Given any real numbers α and β, we refer to the homogeneous second order linear equation

$$x^2 y'' + \alpha x y' + \beta y = 0 \tag{2.10.1}$$

as **Euler's Equation**. We note that Equation (2.10.1) admits exactly one singular point $x_0 = 0$ since the real quadratic polynomial $P(x) = x^2$ has a repeated real root at $x_0 = 0$. Every other real number is an ordinary point of the equation. We will construct in this section the general solution of Euler's Equations for all real numbers $x > 0$. Observe that if $y = x^r$ for some nonzero real number r, then we may deduce a necessary condition for which $y = x^r$ is a solution of Equation (2.10.1) as follows. By the Power Rule, we have that $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$ so that

$$0 = x^2 y'' + \alpha x y' + \beta y = x^2[r(r-1)x^{r-2}] + \alpha x(rx^{r-1}) + \beta x^r = [r^2 + (\alpha - 1)r + \beta]x^r$$

if and only if $r^2 + (\alpha - 1)r + \beta = 0$ since we have assumed that $x > 0$. Consequently, if r is a root of

$$r^2 + (\alpha - 1)r + \beta = 0, \tag{2.10.2}$$

then $y = x^r$ is a solution of Equation (2.10.1). By analogy to the characteristic equation we studied for second order linear equations with constant coefficients, we distinguish Equation (2.10.2) as the

Euler characteristic equation. Like before, the three cases for this equation are determined by the discriminant $(\alpha-1)^2-4\beta$ of the quadratic polynomial $r^2+(\alpha-1)r+\beta$. Quite like the ordinary characteristic equation, the solutions of Euler's Equation are derived for each of these three distinct cases in much the same way. We summarize the results of this section in the following theorem.

Theorem 2.10.1 (Solutions of Euler's Equation)**.** *Consider Euler's Equation*

$$x^2y''+\alpha xy'+\beta=0$$

defined for some real numbers α and β. Every fundamental set of solutions of this second order linear ordinary differential equation depends on the discriminant $(\alpha-1)^2-4\beta$ of the Euler characteristic equation $r^2+(\alpha-1)r+\beta=0$. Explicitly, the (possibly complex) roots of this equation are

$$r_1=\frac{-(\alpha-1)+\sqrt{(\alpha-1)^2-4\beta}}{2} \text{ and } r_2=\frac{-(\alpha-1)-\sqrt{(\alpha-1)^2-4\beta}}{2}.$$

Even more, the general solution of Euler's Equation is completely determined by the following cases.

1.) *Given that $(\alpha-1)^2-4\beta>0$, the general solution of Euler's Equation is*

$$y=\phi(x)=c_1x^{r_1}+c_2x^{r_2}.$$

2.) *Given that $(\alpha-1)^2-4\beta<0$, the roots of the Euler characteristic equation are the complex conjugates $r_1=\lambda+\mu i$ and $r_2=\lambda-\mu i$, and the general solution of Euler's Equation is*

$$y=\phi(x)=c_1x^{\lambda}\cos[\mu\ln(x)]+c_2x^{\lambda}\sin[\mu\ln(x)].$$

3.) *Given that $(\alpha-1)^2-4\beta=0$, there is one root $r_1=-(\alpha-1)/2$ (with multiplicity two) of the Euler characteristic equation, and the general solution of Euler's Equation is*

$$y=\phi(x)=c_1x^r+c_2x^r\ln(x).$$

Proof. We have already established in the exposition preceding the statement of the theorem that the discriminant $(\alpha-1)^2-4\beta$ of the Euler characteristic equation induces the aforementioned three cases for the general solution of Euler's Equation. By the Quadratic Formula, the roots of the Euler characteristic equation are verified. Considering that the solutions of a second order linear ordinary differential equation are unique (up to coefficients) by the Fundamental Theorem of Second Order Linear Ordinary Differential Equations, it suffices to prove that the pair of real univariate functions $\phi_1(x)$ and $\phi_2(x)$ indicated in each of the above three cases form a fundamental set of solutions.

1.) Each of the real univariate functions $\phi_1(x)=x^{r_1}$ and $\phi_2(x)=x^{r_2}$ provides a solution of Euler's Equation by construction. Even more, the Wronskian of x^{r_1} and x^{r_2} is given by

$$W(x^{r_1},x^{r_2})=x^{r_1}(r_2x^{r_2-1})-(r_1x^{r_1-1})x^{r_2}=(r_2-r_1)x^{r_1+r_2-1}.$$

Considering that r_1 and r_2 are distinct roots of the Euler characteristic equation by hypothesis that $(\alpha-1)^2-4\beta>0$, it follows that the Wronskian of x^{r_1} and x^{r_2} is nonzero by assumption that $x>0$. We conclude by Theorem 2.2.8 that the general solution of Euler's Equation is

$$y=\phi(x)=c_1\phi(x)+c_2\phi(x)=c_1x^{r_1}+c_2x^{r_2}.$$

2.) Crucially, we note that if $(\alpha-1)^2-4\beta<0$, then $4\beta-(\alpha-1)^2>0$ so that $\lambda=-(\alpha-1)/2$ and $\mu=\sqrt{4\beta-(\alpha-1)^2}$ are real numbers. Even more, by the Quadratic Formula, it follows that the distinct complex roots of the Euler characteristic equation are $r_1=\lambda+\mu i$ and $r_2=\lambda-\mu i$. Consequently, the complex univariate functions $\gamma_1(x)=x^{\lambda+\mu i}$ and $\gamma_2(x)=x^{\lambda-\mu i}$ provide a pair of solutions of Euler's Equation by construction; however, we seek real solutions of Euler's Equation. We have previously defined the complex exponential function

$$e^{(\alpha+\beta i)x}=e^{\alpha x}\cos(\beta x)+ie^{\alpha x}\sin(\beta x).$$

By analogy to the real exponential function, we define a complex logarithmic function

$$x^{\alpha+\beta i}=\exp[\ln(x^{\alpha+\beta i})]=\exp[(\alpha+\beta i)\ln(x)]=e^{\alpha\ln(x)}\cos[\beta\ln(x)]+ie^{\alpha x}\sin[\beta\ln(x)].$$

Considering that $e^{\alpha\ln(x)}=\exp[\alpha\ln(x)]=\exp[\ln(x^\alpha)]=x^\alpha$ for any pair of real numbers x and α, we obtain at last a definition for the complex exponential function $x^{\alpha+\beta i}$.

$$x^{\alpha+\beta i}=x^\alpha\cos[\beta\ln(x)]+ix^\alpha\sin[\beta\ln(x)]$$

By the Principle of Superposition, for any real numbers c_1 and c_2, we have that

$$\phi_1(x)=\frac{\gamma_1(x)+\gamma_2(x)}{2}=x^\lambda\cos[\mu\ln(x)] \text{ and}$$

$$\phi_2(x)=\frac{\gamma_1(x)-\gamma_2(x)}{2i}=x^\lambda\sin[\mu\ln(x)]$$

are real univariate solutions of Euler's Equation. Check that their Wronskian is nonzero.

3.) Last, we will assume that $r_1=-(\alpha-1)/2$ is a root of the characteristic equation of multiplicity two. Consequently, the Euler characteristic equation can be written as $(r-r_1)^2=0$. Consider the real bivariate function $f(x,r)=(r-r_1)^2x^r=x^2y''+\alpha xy'+\beta y$. We have that

$$f_r(x,r)=\frac{\partial}{\partial r}f(x,r)=\frac{\partial}{\partial r}[(r-r_1)^2x^r]=2(r-r_1)x^r+(r-r_1)^2x^r\ln(x)$$

by the Product Rule, from which it follows that $f_r(x,r_1)=0$. By Clairaut's Theorem, we may interchange the order of differentiation between the variables x and r to find that

$$\begin{aligned}x^2\frac{d^2}{dx^2}[x^r\ln(x)]+\alpha x\frac{d}{dx}[x^r\ln(x)]+\beta x^r\ln(x)&=x^2\frac{d^2}{dx^2}\left[\frac{\partial}{\partial r}x^r\right]+\alpha x\frac{d}{dx}\left[\frac{\partial}{\partial r}x^r\right]+\beta\frac{\partial}{\partial r}x^r\\&=\frac{\partial}{\partial r}\left(x^2\frac{d^2}{dx^2}x^r+\alpha x\frac{d}{dx}x^r+\beta x^r\right)\\&=\frac{\partial}{\partial r}(x^2y''+\alpha xy'+\beta y)\\&=f_r(x,r).\end{aligned}$$

Considering that $f_r(x,r_1)=0$ by a previous calculation, it follows that $x^{r_1}\ln(x)$ is a solution of Euler's Equation. Consequently, we obtain a fundamental set of solutions $\phi_1(x)=x^{r_1}$ and $\phi_2(x)=x^{r_1}\ln(x)$ for Euler's Equation since one can check that their Wronskian is nonzero. □

Chapter 3

Other Solutions of Ordinary Differential Equations

Generalizing from second order linear ordinary differential equations to higher order linear ordinary differential equations is delightfully simple: indeed, many of the techniques to solve linear ordinary differential equations of arbitrary order $n \geq 2$ are analogous to those developed in our study of second order linear equations in Chapter 2. Consequently, we seek at present to provide different strategies for solving nth order linear ordinary differential equations for any integer $n \geq 2$. Both algebraic and numerical methods are developed with an emphasis on equations arising in physics.

3.1 Laplace and Inverse Laplace Transforms

Consider any real univariate function $f(x)$ that is defined for all real numbers $x > a$ for some real number a except at countably infinitely many points. We define the **improper integral**

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx \tag{3.1.1}$$

of $f(x)$ on the interval (a, ∞) as the limit of the definite integrals of $f(x)$ over the intervals (a, b) as b tends to ∞. Granted that these definite integrals exist and their limit as b tends to ∞ exists, we say that the improper integral in $f(x)$ **converges**; otherwise, the integral is said to **diverge**.

Example 3.1.1. Compute all values of p for which the following indefinite integral converges.

$$\int_0^\infty e^{px}\,dx$$

Solution. Certainly, if $p = 0$, then the integral diverges because $e^{px} = 1$ in this case. Consequently, we may assume that p is nonzero, hence the antiderivative of e^{px} is determined by the Chain Rule.

$$\int_0^\infty e^{px}\,dx = \lim_{b \to \infty} \int_0^b e^{px}\,dx = \lim_{b \to \infty} \left[\frac{e^{px}}{p}\right]_0^b = \lim_{b \to \infty} \frac{1}{p}(e^{bp} - 1) = \begin{cases} \text{diverges} & \text{if } p \geq 0 \text{ and} \\ -\dfrac{1}{p} & \text{if } p < 0. \end{cases} \quad \diamond$$

Example 3.1.2. Compute all values of p for which the following indefinite integral converges.

$$\int_1^\infty x^{-p}\,dx$$

Solution. Like before, we must consider the case that $p = 1$ separately: indeed, if $p = 1$, then

$$\int_1^\infty x^{-1}\,dx = \lim_{b\to\infty}\int_1^b x^{-1}\,dx = \lim_{b\to\infty}[\ln(x)]_1^b = \lim_{b\to\infty}\ln(b) = \infty.$$

On the other hand, if $p \neq 1$, then the antiderivative of x^{-p} is determined by the Power Rule.

$$\int_1^\infty x^{-p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty}\left(\frac{b^{1-p}-1}{1-p}\right) = \begin{cases}\text{diverges} & \text{if } p \leq 1 \text{ and} \\ -\dfrac{1}{1-p} & \text{if } p > 1.\end{cases} \qquad \diamond$$

Crucially, to guarantee the convergence of the integral (3.1.1), we may assume that the integrand $f(x)$ is **piecewise continuous** on the open interval (a, b) for each real number $b > a$. Explicitly, we seek a **partition** of (a, b) into finitely many points $a = x_0 < x_1 < \cdots < x_n = b$ such that

1.) $f(x)$ is continuous on each open subinterval (x_i, x_{i+1}) for each integer $0 \leq i \leq n-1$ and

2.) the left- and right-hand limits of $f(x)$ in each open subinterval (x_i, x_{i+1}) are finite.

Put another way, it follows that $f(x)$ is piecewise continuous on an open interval (a, b) if and only if it is continuous on (a, b) except possibly at a countably infinite number of points at which there may be jump discontinuities. We will not endeavor to prove this; however, it follows from the fact that a function that is continuous on an open interval is integrable on that open interval and the definite integral of an integrable function does not depend on countably infinite jump discontinuities.

We refer the reader to Section 0.15 for more information and examples on improper integration; however, due to its importance in the forthcoming definition of an integral transform, we recall one of the most pleasant features of improper integration in the following comparison theorem.

Theorem 3.1.3 (Comparison Theorem for Improper Integrals)**.** *Consider any real univariate function $f(x)$ that is piecewise continuous for all real numbers $x > a$.*

1.) *Given any real univariate function $g(x)$ such that $|f(x)| \leq g(x)$ for all real numbers $x \geq M$ for some positive real number M, if $\int_M^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.*

2.) *Given any real univariate function $g(x)$ such that $f(x) \geq g(x) \geq 0$ for all real numbers $x \geq M$ for some positive real number M, if $\int_M^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.*

Consequently, the convergence of $\int_a^\infty f(x)\,dx$ is determined by the convergence of $\int_M^\infty g(x)\,dx$.

Generally, we define an **integral transform** $F(s)$ with **kernel** $K(s, t)$ via the definite integral

$$F(s) = \int_a^b K(s,t)f(t)\,dt$$

for any (possibly infinite) real numbers a and b for which the definite integral exists. Crucially, $F(s)$ is a function that depends on a, b, and s; however, we will typically assume that a and b are fixed so that $F(s)$ depends only on s, i.e., it is a real univariate function. Of all possible integral transforms, we seek to study the **Laplace transform** with kernel $K(s, t) = e^{-st}$ over the open interval $(0, \infty)$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt.$$

We will come to find in short order that the Laplace transform is an invaluable tool in the study of linear ordinary differential equations because it allows us to convert from a differential equation to a rational equation. Before we arrive at this invaluable observation, we require the following.

Theorem 3.1.4 (Existence of the Laplace Transform)**.** *Consider any real univariate function $f(t)$ that is piecewise continuous on the closed interval $[0, b]$ for each real number $b \geq 0$. Given that for each real number M, there exist real numbers a and C that do not depend on t such that $|f(t)| \leq Ce^{at}$ for all real numbers $t \geq M$, the Laplace transform $\mathcal{L}\{f(t)\}$ exists for all real numbers $s \geq a$.*

Example 3.1.5. Compute the Laplace transform of $f(t) = a$ for any real number a.

Solution. Clearly, the constant function $f(t) = a$ satisfies the hypotheses of Theorem 3.1.4 because e^t is increasing, so its Laplace transform exists: indeed, we have that

$$\mathcal{L}\{a\} = \int_0^\infty ae^{-st}\,dt = a\int_0^\infty e^{-st}\,dt = \left[-\frac{a}{s}e^{-st}\right]_0^\infty = \frac{a}{s} \text{ for all real numbers } s > 0.$$

Crucially, the domain of $\mathcal{L}\{a\}$ consists of all positive real numbers. One can readily verify that for any non-positive real number s, the improper integral that defines $\mathcal{L}\{a\}$ diverges. ⋄

Example 3.1.6. Compute the Laplace transform of $f(t) = e^{at}$ for any real number a.

Solution. Our exponential function is trivially bounded by itself for all real numbers, hence

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty = \frac{1}{s-a} \text{ for all real numbers } s > a.$$

Once again, the domain of $\mathcal{L}\{e^{at}\}$ is critically important: for any real number $s \leq a$, it follows that $a - s \geq 0$ so that the improper integral that defines $\mathcal{L}\{e^{at}\}$ diverges. ⋄

Example 3.1.7. Compute the Laplace transform of $f(t) = \cos(at)$ for any real number a.

Solution. Considering that $\cos(at)$ oscillates between -1 and 1, it is eventually bounded by any exponential function, so the Laplace transform of $\cos(at)$ exists. We use integration by parts.

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= \int_0^\infty e^{-st}\cos(at)\,dt \\
&= \left[-\frac{1}{s}e^{-st}\cos(at)\right]_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)\,dt && (u = \cos(at) \text{ and } dv = e^{-st}\,dt) \\
&= \frac{1}{s} - \frac{a}{s}\left(\left[-\frac{1}{s}e^{-st}\sin(at)\right]_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\,dt\right) && (u = \sin(at) \text{ and } dv = e^{-st}\,dt) \\
&= \frac{1}{s} - \frac{a^2}{s^2}\mathcal{L}\{\cos(at)\} && \text{(Recognize the integral as a Laplace transform.)}
\end{aligned}$$

By solving for the Laplace transform, we obtain $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2 + a^2}$ for all real numbers $s > 0$. ⋄

Crucially, the Laplace transform $\mathcal{L}\{f(t)\}$ is a **linear transformation** (or a **linear operator**) from the real vector space of real univariate functions to itself. Explicitly, for any real univariate functions $f(t)$ and $g(t)$ for which the Laplace transform is defined and any real number C, we have

$$\begin{aligned}\mathcal{L}\{Cf(t)+g(t)\} &= \int_0^\infty e^{-st}[Cf(t)+g(t)]\,dt \\ &= \int_0^\infty e^{-st}[Cf(t)]\,dt + \int_0^\infty e^{-st}g(t)\,dt \\ &= C\int_0^\infty e^{-st}f(t)\,dt + \int_0^\infty e^{-st}g(t) \\ &= C\mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.\end{aligned}$$

Consequently, the Laplace transform is **linear**: it commutes with addition and scalar multiplication. Considering that the Laplace transform is defined as an integral, it is not surprising that it is linear because the integral itself is linear; however, we will come to appreciate this property even more.

Example 3.1.8. Compute the Laplace transform of $f(t) = 3e^{-2t} - \cos(t)$.

Solution. By the linearity of the Laplace transform, we have that

$$\mathcal{L}\{3e^{-2t} - \cos(t)\} = 3\mathcal{L}\{e^{-2t}\} - \mathcal{L}\{\cos(t)\} = \frac{3}{s+2} - \frac{s}{s^2+1} \text{ for all real numbers } s > 0. \qquad \diamond$$

Example 3.1.9. Compute the Laplace transform of $f(t) = \sinh(at)$ for any real number a.

Solution. By the linearity of the Laplace transform, we have that

$$\mathcal{L}\{\sinh(at)\} = \mathcal{L}\left\{\frac{1}{2}e^{at} - \frac{1}{2}e^{-at}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\} = \frac{1}{2(s-a)} - \frac{1}{2(s+a)}.$$

for all real numbers $s > \max\{a, -a\} = |a|$. Getting a common denominator, we obtain

$$\mathcal{L}\{\sinh(at)\} = \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{(s+a)-(s-a)}{2(s-a)(s+a)} = \frac{a}{s^2-a^2} \text{ for } s > |a|. \qquad \diamond$$

Example 3.1.10. Compute the Laplace transform of $f(t) = t^2 - t + 1$.

Solution. By the linearity of the Laplace transform, we have that

$$\mathcal{L}\{t^2 - t + 1\} = \mathcal{L}\{t^2\} - \mathcal{L}\{t\} + \mathcal{L}\{1\}.$$

Consequently, it suffices to compute $\mathcal{L}\{t^2\}$ and $\mathcal{L}\{t\}$. We use integration by parts with $u = t^2$.

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2e^{-st}\,dt = \left[-\frac{1}{s}t^2e^{-st}\right]_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}\,dt = \frac{2}{s}\mathcal{L}\{t\}.$$

Crucially, we obtain $\mathcal{L}\{t^2\}$ as a product of a rational function and $\mathcal{L}\{t\}$, so it suffices to determine $\mathcal{L}\{t\}$. Once again, we will employ the technique of integration by parts with $u = t$.

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,dt = \left[-\frac{1}{s}te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \left[-\frac{1}{s^2}e^{-st}\right]_0^\infty = \frac{1}{s^2}.$$

We conclude by our recursive formula for $\mathcal{L}\{t^2\}$ and the linearity of the Laplace transform that

$$\mathcal{L}\{t^2 - t + 1\} = \mathcal{L}\{t^2\} - \mathcal{L}\{t\} + \mathcal{L}\{1\} = \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{s} \text{ for all real numbers } s > 0. \qquad \diamond$$

Example 3.1.11. Compute the Laplace transform of $f(t) = t^n$ for any integer $n \geq 0$.

Solution. Considering the recursive formula of Example 3.1.10, we surmise that

$$\mathcal{L}\{t^{n+1}\} = \frac{n+1}{s}\mathcal{L}\{t^n\}$$

for any integer $n \geq 0$. Consequently, using our $\mathcal{L}\{1\}$ from Example 3.1.5, we conclude that

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\} = \frac{n(n-1)}{s^2}\mathcal{L}\{t^{n-2}\} = \cdots = \frac{n!}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^{n+1}} \text{ for all real numbers } s > 0. \qquad \diamond$$

One of the most important properties that a linear transformation may satisfy is **injectivity**. Explicitly, we say that a function $f(x)$ is **injective** if and only if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ for all x-values in the domain of $f(x)$. Considering that the Laplace transform is a function, we can determine if it is injective; moreover, the fact that $\mathcal{L}$ is a linear operator ensures that $\mathcal{L}$ is injective if and only if $\mathcal{L}\{f(t)\} = 0$ implies that $f(t) = 0$ for all t-values such that $f(t)$ lies in the domain of $\mathcal{L}\{f(t)\}$. By applying the definition of $\mathcal{L}\{f(t)\}$ as an improper integral, it follows that if

$$0 = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s),$$

then we must have that $F(s) = 0$ for all valid s-values. We can check this for a given function $f(t)$; however, the injectivity of the Laplace transform is in general guaranteed by Lerch's Theorem. We will therefore not endeavor to prove this fact in general; we will simply seek to determine the inverse Laplace transform of a function whose Laplace transform is known (or can be computed). Primarily, for our considerations, this involves merely recognizing the function $f(t)$ for which $\mathcal{L}\{f(t)\} = F(s)$.

Example 3.1.12. Compute the inverse Laplace transform of $F(s) = \frac{a}{s}$ for any real number a.

Solution. We note that $\mathcal{L}\{a\} = \frac{a}{s}$ for $s > 0$ so that $\mathcal{L}^{-1}\left\{\frac{a}{s}\right\} = a$ for $t \geq 0$. $\diamond$

Example 3.1.13. Compute the inverse Laplace transform of $F(s) = \dfrac{1}{s-a}$ for any real number a.

Solution. We note that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ for $s > a$ so that $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ for $t \geq 0$. $\diamond$

Example 3.1.14. Compute the inverse Laplace transform of $F(s) = \dfrac{s}{s^2+a^2}$ for a real number a.

Solution. We note that $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$ for $s > 0$ and $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos(at)$ for $t \geq 0$. $\diamond$

3.2 Laplace Transform and Initial Value Problems

We have determined thus far that the Laplace transform of a real univariate function $f(t)$ is a linear operator from the real vector space of real univariate functions to itself. Generally, the existence of the Laplace transform $\mathcal{L}\{f(t)\}$ of a real univariate function $f(t)$ defined for all real numbers $t \geq 0$ is completely determined by Theorem 3.1.4. We will restrict our attention to those real univariate functions whose Laplace transform is well-defined. Each example of the Laplace transform we have encountered thus far has been a rational function of a real variable s. Even more, we will illustrate in this section that the Laplace transform can be used to convert a linear ordinary differential equation into a rational equation that can be solved algebraically; the solution of the differential equation can be obtained as the inverse Laplace transform of the resulting rational function. Before we outline the general technique, we require the following distributive law for the Laplace transform.

Proposition 3.2.1 (Laplace Transform and the Derivative)**.** *Consider any real univariate function $f(t)$ such that for each real number M, there exist real numbers a and C that do not depend on t such that $|f(t)| \leq Ce^{at}$ for all real numbers $t \geq M$. Given that $f'(t)$ is piecewise continuous on the closed interval $[0, b]$ for each real number $b \geq 0$, the Laplace transform of $f'(t)$ is given by*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \text{ for all real numbers } s > a.$$

Proof. We illustrate the proof because it is fundamental in understanding the formula. Given any real number $b \geq 0$, let $t_1, t_2, \ldots, t_n$ denote the real numbers in the closed interval $[0, b]$ for which $f'(t)$ is discontinuous. By hypothesis that $f'(t)$ is piecewise continuous on the closed interval $[0, b]$, it follows that $t_1, t_2, \ldots, t_n$ are jump discontinuities, hence we have that

$$\int_0^b e^{-st} f'(t)\, dt = \sum_{i=0}^n \int_{t_i}^{t_{i+1}} e^{-st} f'(t)\, dt$$

for $t_0 = 0$ and $t_{n+1} = b$. Each of the definite integrals over the open intervals (t_i, t_{i+1}) exists because $f'(t)$ is continuous on each of these intervals. Even more, they can be determined explicitly using integration by parts with $u = e^{-st}$ and $dv = f'(t)\, dt$. Carrying this out yields that that

$$\int_{t_i}^{t_{i+1}} e^{-st} f'(t)\, dt = \left[e^{-st} f(t)\right]_{t_i}^{t_{i+1}} + s\int_{t_i}^{t_{i+1}} e^{-st} f(t)\, dt = e^{-st_{i+1}} f(t_{i+1}) - e^{-st_i} f(t_i) + s\int_{t_i}^{t_{i+1}} e^{-st} f(t)\, dt$$

for each integer $0 \leq i \leq n+1$. By summing these definite integrals for each integer $0 \leq i \leq n+1$, we find that the terms involving $e^{-st_i} f(t_i)$ cancel for each integer $1 \leq i \leq n$, leaving only

$$\int_0^b e^{-st} f'(t)\, dt = e^{-sb} f(b) - f(0) + s\sum_{i=0}^n \int_{t_i}^{t_{i+1}} e^{-st} f(t)\, dt = e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)\, dt.$$

Considering that $|f(t)| \leq Ce^{at}$ for all real numbers $t \geq M$, it follows that

$$\mathcal{L}\{f'(t)\} = \lim_{b\to\infty} \int_0^b e^{-st} f'(t)\, dt = \lim_{b\to\infty} \left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)\, dt\right] = s\mathcal{L}\{f(t)\} - f(0)$$

for all real numbers $s > a$ since the limit involving $e^{-sb} f(b)$ converges to 0 if $s > a$. □

Corollary 3.2.2 (Recursive Formula for the Laplace Transform of a Derivative)**.** *Consider any real univariate function $f(t)$ such that for each real number M, there exist real numbers a and C that do not depend on t such that $|f^{(k)}(t)| \leq Ce^{at}$ for all real numbers $t \geq M$ for each integer $0 \leq k \leq n-1$. Given that $f^{(n)}(t)$ is piecewise continuous on the closed interval $[0, b]$ for each real number $b \geq 0$, the Laplace transform of $f^{(n)}(t)$ can be obtained recursively according to the following formula.*

$$\mathcal{L}\{f^{(n)}(t)\} = s\mathcal{L}\{f^{(n-1)}(t)\} - f^{(n-1)}(0) \text{ for all real numbers } s > a$$

Explicitly, the Laplace transform of $f^{(n)}(t)$ is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \text{ for all real numbers } s > a.$$

Example 3.2.3. Use the Laplace transform to solve the initial value problem.

$$y'' + 3y' + 2y = 0 \text{ with } y(0) = 1 \text{ and } y'(0) = 0$$

Solution. Using the linearity of the Laplace transform as well as Corollary 3.2.2, we find that

$$\begin{aligned}
\mathcal{L}\{y'' + 3y' + 2y\} &= \mathcal{L}\{0\} \\
\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= 0 \\
s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3[s\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{y\} &= 0 \\
(s^2 + 3s + 2)\mathcal{L}\{y\} - s - 3 &= 0 \\
\mathcal{L}\{y\} &= \frac{s+3}{s^2 + 3s + 2}.
\end{aligned}$$

Consequently, in order to determine $y(t)$, it suffices to determine the inverse Laplace transform of $F(s) = \mathcal{L}\{y\}$. We achieve this via partial fraction decomposition on $s^2 + 3s + 2 = (s+1)(s+2)$.

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$s + 3 = A(s+2) + B(s+1)$$

By plugging in $s = -1$, we find that $A = 2$. Likewise, if $s = -2$, then $-B = 1$ so that $B = -1$. We conclude by the linearity of the Laplace transform that $\mathcal{L}\{y\}$ is given by

$$\mathcal{L}\{y\} = \frac{s+2}{s^2 + 3s + 2} = 2 \cdot \frac{1}{s+1} - \frac{1}{s+2} = 2\mathcal{L}\{e^{-t}\} - \mathcal{L}\{e^{-2t}\} = \mathcal{L}\{2e^{-t} - e^{-2t}\}.$$

Considering that the Laplace transform is injective, it follows that $y(t) = 2e^{-t} - e^{-2t}$. ◇

Unfortunately, we will come to find in the next example that if the characteristic equation of the second order linear equation has repeated roots, then our current toolkit of Laplace transforms is inadequate. Explicitly, if the characteristic equation is $(r - r_1)^2 = 0$ for some real number r_1, then the partial fraction decomposition required to determine the inverse Laplace transform typically results in a proper rational function whose numerator is constant and whose denominator is of the form $(s - r_1)^2$. Before proceeding with the remaining examples, we resolve this issue as follows.

Example 3.2.4. Compute the Laplace transform of $t^n e^{at}$ for any integer $n \geq 0$ and real number a.

Solution. By definition of the Laplace transform, we seek to determine the improper integral

$$\mathcal{L}\{t^n e^{at}\} = \int_0^\infty e^{-st} t^n e^{at}\, dt = \int_0^\infty t^n e^{-(s-a)t}\, dt.$$

Crucially, we may recognize this as none other than the Laplace transform of t^n shifted to the left by a units in the sy-plane. Explicitly, if $F(s) = \mathcal{L}\{t^n\}$, then for any real number $s > a$, we have

$$F(s-a) = \int_0^\infty t^n e^{-(s-a)t}\, dt = \mathcal{L}\{t^n e^{at}\}.$$

Considering that we derived $F(s)$ as a rational function in Example 3.1.11, we conclude that

$$\mathcal{L}\{t^n e^{at}\} = F(s-a) = \frac{n!}{(s-a)^{n+1}} \text{ for all real numbers } s > a. \qquad \diamond$$

Proposition 3.2.5 (Shifting Property of the Laplace Transform)**.** *Given any real univariate function $f(t)$ for which the Laplace transform is defined, the Laplace transform of $e^{at} f(t)$ is given by the Laplace transform of $f(t)$ shifted to the left by a units in the sy-plane. Explicitly, if $\mathcal{L}\{f(t)\} = F(s)$ for all real numbers $s > C$, then $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ for all real numbers $s > a + C$.*

Proof. By hypothesis, the following improper integral converges to $F(s)$ for all real numbers $s > C$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Consequently, for all real numbers $s - a > C$, the following improper integral converges.

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{-st} e^{at} f(t)\, dt = \mathcal{L}\{e^{at} f(t)\} \qquad \square$$

We will come to find that the Shifting Property of the Laplace Transform provides an indispensable tool in the theory of initial value problems involving linear ordinary differential equations. Be sure to notice its importance in the following pair of examples involving initial value problems with both homogeneous and non-homogeneous second order linear ordinary differential equations.

Example 3.2.6. Use the Laplace transform to solve the initial value problem.

$$y'' - 4y' + 4y = 0 \text{ with } y(0) = 1 \text{ and } y'(0) = 1$$

Solution. Using the linearity of the Laplace transform as well as Corollary 3.2.2, we find that

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 4[s\mathcal{L}\{y\} - y(0)] + 4\mathcal{L}\{y\} = 0$$

$$(s^2 - 4s + 4)\mathcal{L}\{y\} - s + 3 = 0$$

$$\mathcal{L}\{y\} = \frac{s-3}{s^2 - 4s + 4}.$$

Consequently, in order to determine $y(t)$, it suffices to determine the inverse Laplace transform of $F(s) = \mathcal{L}\{y\}$. We achieve this via partial fraction decomposition on $s^2 - 4s + 4 = (s-2)^2$.

$$\frac{s-3}{(s-2)^2} = \frac{A}{s-2} + \frac{B}{(s-2)^2}$$

$$s - 3 = A(s-2) + B$$

By plugging in $s = 2$, we find that $B = -1$ so that $A(s-2) = s-2$, from which we find that $A = 1$. We conclude by the linearity of the Laplace transform that $\mathcal{L}\{y\}$ is given by

$$\mathcal{L}\{y\} = \frac{s-3}{s^2-4s+4} = \frac{1}{s-2} - \frac{1}{(s-2)^2} = \mathcal{L}\{e^{2t}\} - \mathcal{L}\{te^{2t}\} = \mathcal{L}\{e^{2t} - te^{2t}\}.$$

Explicitly, we compute the Laplace transform $\mathcal{L}\{(s-2)^{-2}\} = t^2 e^{2t}$ via the Shifting Property of the Laplace Transform. By the injectivity of the Laplace transform, we obtain $y(t) = e^{2t} - te^{2t}$. ◇

Example 3.2.7. Use the Laplace transform to solve the initial value problem.

$$y'' + 2y' + y = 4e^{-t} \text{ with } y(0) = 2 \text{ and } y'(0) = -1$$

Solution. Using the linearity of the Laplace transform as well as Corollary 3.2.2, we find that

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{4e^{-t}\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 4\mathcal{L}\{e^{-t}\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2[s\mathcal{L}\{y\} - y(0)] + \mathcal{L}\{y\} = \frac{4}{s+1}$$

$$(s^2 + 2s + 1)\mathcal{L}\{y\} - 2s - 3 = \frac{4}{s+1}$$

$$\mathcal{L}\{y\} = \frac{4}{(s+1)(s^2+2s+1)} + \frac{2s+3}{s^2+2s+1}.$$

Consequently, in order to determine $y(t)$, it suffices to determine the inverse Laplace transform of $F(s) = \mathcal{L}\{y\}$. We achieve this via partial fraction decomposition on $(s+1)(s^2+2s+1) = (s+1)^3$.

$$\frac{4}{(s+1)(s^2+2s+1)} + \frac{2s+3}{s^2+2s+1} = \frac{(2s+3)(s+1)+4}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$2s^2+5s+7 = A(s+1)^2 + B(s+1) + C$$

By plugging in $s=-1$, we find that $C=4$ so that $2s^2+5s+3 = A(s+1)^2+B(s+1)$. Comparing coefficients of the quadratic polynomial on the left-hand side with the right-hand side, we find that

$$2s^2+5s+3 = A(s+1)^2 + B(s+1) = As^2 + (2A+B)s + (A+B)$$

so that $A=2$ and $B=1$. We conclude by the linearity of the Laplace transform that $\mathcal{L}\{y\}$ satisfies

$$\begin{aligned}\mathcal{L}\{y\} &= \frac{(2s+3)(s+1)+4}{(s+1)^3} \\ &= 2\cdot\frac{1}{s+1} + \frac{1}{(s+1)^2} + 4\cdot\frac{1}{(s+1)^3} \\ &= 2\mathcal{L}\{e^{-t}\} + \mathcal{L}\{t^2e^{-t}\} + 2\mathcal{L}\{t^2e^{-t}\} \\ &= \mathcal{L}\big\{2e^{-t} + te^{-t} + 2t^2e^{-t}\big\}.\end{aligned}$$

Explicitly, we compute the Laplace transforms of $(s+1)^{-2}$ and $(s+1)^{-3}$ via the Shifting Property of the Laplace Transform; thus, we obtain the desired solution $y(t) = 2e^{-t} + te^{-t} + 2t^2e^{-t}$. ◇

Example 3.2.8. Use the Laplace transform to solve the initial value problem.

$$y^{(4)} - y = 0 \text{ with } y(0)=1,\ y'(0)=0,\ y''(0)=1, \text{ and } y'''(0)=0$$

Solution. Using the linearity of the Laplace transform as well as Corollary 3.2.2, we find that

$$\begin{aligned}\mathcal{L}\{y^{(4)} - y\} &= \mathcal{L}\{0\} \\ \mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} &= 0 \\ s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - \mathcal{L}\{y\} &= 0 \\ (s^4-1)\mathcal{L}\{y\} - s^3 - s &= 0 \\ \mathcal{L}\{y\} &= \frac{s^3+s}{s^4-1} = \frac{s}{s^2-1}.\end{aligned}$$

Consequently, in order to determine $y(t)$, it suffices to determine the inverse Laplace transform of $F(s) = \mathcal{L}\{y\}$. We achieve this via partial fraction decomposition on $s^2 - 1$.

$$\frac{s}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$$

$$s = A(s-1) + B(s+1)$$

By plugging in $s = -1$, we find that $-2A = -1$ so that $A = \frac{1}{2}$. Likewise, if $s = 1$, then $2B = 1$ so that $B = \frac{1}{2}$. We conclude by the linearity of the Laplace transform that $\mathcal{L}\{y\}$ is given by

$$\mathcal{L}\{y\} = \frac{s}{s^2-1} = \frac{1}{2} \cdot \frac{1}{s-1} + \frac{1}{2} \cdot \frac{1}{s+1} = \frac{1}{2}\mathcal{L}\{e^t\} + \frac{1}{2}\mathcal{L}\{e^{-t}\} = \mathcal{L}\left\{\frac{1}{2}e^t + \frac{1}{2}e^{-t}\right\}.$$

Considering that the Laplace transform is injective, it follows that $y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t)$. ◇

Before we conclude this section, we remark on the novelty of Example 3.2.8: we had not prior to this excursion solved a linear ordinary differential equation of order exceeding two! Crucially, the Recursive Formula for the Laplace Transform of a Derivative provides a brand new technique to solve many initial value problems involving linear ordinary differential equations of order $n \geq 1$.

3.3 Step Functions

We commence in this section our discussion of ordinary differential equations in the context of their physical applications. We will come to find that the Laplace transform is especially useful in these situations because it behaves well with respect to functions that admit countably infinitely many jump discontinuities. Each of these such functions can be assembled from a very simple function $u_c(t)$ called the **unit step function** or the **Heaviside function** centered at $t = c$ defined by

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \text{ and} \\ 1 & \text{if } t \geq c. \end{cases}$$

Crucially, the unit step function centered at $t = c$ is a piecewise continuous function with a single jump discontinuity at $t = c$. Pictorially, its graph consists of two horizontal lines: the first line extends indefinitely leftward from the open circle at $t = c$ along the x-axis; the second line extends indefinitely rightward from the closed circle at $t = c$ at a height of $y = 1$. Especially interesting are the additive and multiplicative properties of the unit step functions, as illustrated in the following.

Example 3.3.1. Give the following sum of unit step functions as a piecewise function.

$$f(t) = u_{-1}(t) + u_0(t) + u_1(t)$$

Solution. By definition of $u_{-1}(t)$, it follows that $f(t) = 0$ for all real numbers $t < -1$ and $f(t) = 1$ for all real numbers $-1 \leq t < 0$. Likewise, we have that $u_0(t) = 1$ for all real numbers $t \geq 0$, hence we have that $f(t) = 2$ for all real numbers $0 \leq t < 1$. Last, we have that $u_1(t) = 1$ for all real numbers $t \geq 1$ so that $f(t) = 3$ for all real numbers $t \geq 1$. We conclude that $f(t)$ is defined by

$$f(t) = \begin{cases} 0 & \text{if } t < -1, \\ 1 & \text{if } -1 \leq t < 0, \\ 2 & \text{if } 0 \leq t < 1, \text{ and} \\ 3 & \text{if } t \geq 1. \end{cases}$$ ◇

Example 3.3.2. Give the following piecewise function in terms of unit step functions.

$$f(t) = \begin{cases} 2 & \text{if } t < -1, \\ 0 & \text{if } -1 \leq t < 3, \text{ and} \\ 2 & \text{if } t \geq 3 \end{cases}$$

Solution. Considering that $f(t) = 2$ for all real numbers $t < -1$ and $f(t) = 0$ if $-1 \leq t < 3$, we are inclined to believe that $f(t) = 2 - 2u_{-1}(t)$ for all real numbers $t < 3$. Even more, for all real numbers $t \geq 3$, we have that $f(t) = 2$, hence we may add the step function $2u_3(t)$ to achieve this.

$$f(t) = 2 - 2u_{-1}(t) + 2u_3(t)$$ ◇

Example 3.3.3. Give the following piecewise function in terms of unit step functions.

$$f(t) = \begin{cases} \sin(t) & \text{if } t < 0 \\ \cos(t) & \text{if } t \geq 0 \end{cases}$$

Solution. Quite simply, we may take $f(t) = \sin(t) + u_0(t)[\cos(t) - \sin(t)]$. ◇

We will henceforth consider the unit step function $u_c(t)$ as a "switch" centered at $t = c$ in the following sense: for all real numbers $t < c$, the "switch" is turned off, and for all real numbers $t \geq c$, the "switch" is turned on. Consequently, for any real univariate function $f(t)$, the piecewise-defined function $g(t) = u_c(t)f(t)$ may be viewed as the real univariate function that is identically zero for all real numbers $t < c$ and $f(t)$ for all real numbers $t \geq c$. Electrical current is modelled in this way. Considering this perspective, we turn our attention to the behavior of the unit step function $u_c(t)$ with respect to the Laplace transform: if we assume that $c \geq 0$, then by definition, we have that

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt = \frac{1}{s} e^{-cs} \text{ for all real numbers } s > 0 \tag{3.3.1}$$

because $e^{-st}u_c(t) = 0$ for all real numbers $t < c$ and $e^{-st}u_c(t) = e^{-st}$ for all real numbers $t \geq c$. Even more, if we view multiplication of a real univariate function $f(t)$ by $u_c(t)$ as translation to the right by c units, the above calculation suggests an important property of the Laplace transform.

Proposition 3.3.4 (Scaling Property of the Laplace Transform)**.** *Given any real univariate function $f(t)$ for which the Laplace transform is defined, the Laplace transform of $u_c(t)f(t-c)$ is given by the Laplace transform of $f(t)$ scaled by a factor of e^{-cs} in the sy-plane. Explicitly, if $\mathcal{L}\{f(t)\} = F(s)$ for all real numbers $s > a$, then $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$ for all real numbers $s > a$.*

Proof. We seek to exhibit a closed form for the following improper integral on its domain.

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty e^{-st}f(t-c)\,dt$$

Consider the change of variables $u = t - c$ so that $-st = -s(c+u) = -cs - su$. We have that

$$\int_c^\infty e^{-st}f(t-c)\,dt = \int_0^\infty e^{-cs-su}f(u)\,du = e^{-cs}\int_0^\infty e^{-su}f(u)\,du = e^{-cs}\int_0^\infty e^{-st}f(t)\,dt.$$

Crucially, we may recognize the latter improper integral as the Laplace transform of $f(t)$ so that

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s) \text{ for all real numbers } s > a.$$ □

Example 3.3.5. Use the Laplace transform to solve the initial value problem.

$$y'' + 2y' + 2y = \begin{cases} 1 & \text{if } \pi \leq t < 2\pi \text{ and} \\ 0 & \text{otherwise} \end{cases} \quad \text{with } y(0) = 0 \text{ and } y'(0) = 1$$

Solution. We must first define the differential equation in terms of unit step functions. Considering $y'' + 2y' + 2y$ as a unit switch that is off for all real numbers $t < \pi$ and $t \geq 2\pi$, it follows that

$$y'' + 2y' + 2y = u_\pi(t) - u_{2\pi}(t).$$

Using the linearity of the Laplace transform, the Recursive Formula for the Laplace Transform of a Derivative, and Equation 3.3.1, we proceed to evaluate the initial value problem as follows.

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{u_\pi(t) - u_{2\pi}(t)\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{u_\pi(t)\} - \mathcal{L}\{u_{2\pi}(t)\}$$

$$(s^2 + 2s + 2)\mathcal{L}\{y\} - 1 = \frac{1}{s}(e^{-\pi s} - e^{-2\pi s})$$

$$\mathcal{L}\{y\} = (e^{-\pi s} - e^{-2\pi s})\frac{1}{s(s^2 + 2s + 2)} + \frac{1}{s^2 + 2s + 2}$$

We require partial fraction decomposition to determine the form of $\mathcal{L}\{y\}$ so that we may ultimately deduce $y(t)$ using the inverse Laplace transform. Considering that the discriminant of $s^2 + 2s + 2$ is $2^2 - 4(1)(2) < 0$, it follows that $s^2 + 2s + 2$ is an irreducible quadratic polynomial, hence we take

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

for some real numbers A, B, and C. Crucially, we may view the exponential function $e^{-\pi s} - e^{-2\pi s}$ as a coefficient, so it does not affect the partial fraction decomposition. Clearing denominators yields

$$1 = A(s^2 + 2s + 2) + (Bs + C)s$$

so that $2A = 1$ with $s = 0$. We conclude that $A = \frac{1}{2}$. Expanding the quadratic polynomial on the right-hand side, we find that $1 = \big(B + \frac{1}{2}\big)s^2 + (C+1)s + 1$. Comparing coefficients, we must have that $B + \frac{1}{2} = 0$ so that $B = -\frac{1}{2}$ and $C + 1 = 0$ so that $C = -1$. Last, we recognize that $s^2 + 2s + s = (s+1)^2 + 1$ so that we may ultimately express the Laplace transform of $y(t)$ as

$$\begin{aligned}\mathcal{L}\{y\} &= (e^{-\pi s} - e^{-2\pi s})\left(\frac{1}{2} \cdot \frac{1}{s} - \frac{\frac{1}{2}s + 1}{(s+1)^2 + 1}\right) + \frac{1}{(s+1)^2+1} \\ &= (e^{-\pi s} - e^{-2\pi s})\left(\frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s+1}{(s+1)^2+1} - \frac{1}{2} \cdot \frac{1}{(s+1)^2+1}\right) + \frac{1}{(s+1)^2+1}.\end{aligned}$$

Each of the rational functions with an exponential coefficient represents the Laplace transform of a real univariate function of the form $u_c(t)f(t-c)$ for some real number c by the Scaling Property of the Laplace Transform. Even more, each of the rational functions in $s+1$ is the shifted Laplace transform of a real univariate function $g(t)$ by the Shifting Property of the Laplace Transform. Carefully, we recover the inverse Laplace transform of the right-hand side to find that

$$y(t) = \frac{1}{2}[u_\pi(t) - u_{2\pi}(t)]\big[1 - e^{-t}\cos(t) - e^{-t}\sin(t)\big] + e^{-t}\sin(t) \text{ for all real numbers } t \geq 0. \qquad \diamond$$

Example 3.3.6. Use the Laplace transform to solve the initial value problem.

$$y'' + 4y = \begin{cases} \sin(t) & \text{if } 0 \leq t < \pi \text{ and} \\ \sin(t) + \sin(t - \pi) & \text{if } t \geq \pi \end{cases} \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0$$

Solution. Once again, we must recognize the differential equation in terms of unit step functions.

$$y'' + 4y = u_0(t)\sin(t) + u_\pi(t)\sin(t - \pi)$$

We proceed per usual according to the linearity of the Laplace transform and the Recursive Formula for the Laplace Transform of a Derivative to evaluate the initial value problem as follows.

$$\begin{aligned}\mathcal{L}\{y'' + 4y\} &= \mathcal{L}\{u_0(t)\sin(t) + u_\pi(t)\sin(t-\pi)\} \\ \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} &= \mathcal{L}\{u_0(t)\sin(t)\} + \mathcal{L}\{u_\pi(t)\sin(t-\pi)\} \\ (s^2+4)\mathcal{L}\{y\} &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} \\ \mathcal{L}\{y\} &= \frac{1}{(s^2+1)(s^2+4)} + e^{-\pi s}\frac{1}{(s^2+1)(s^2+4)}\end{aligned}$$

We require partial fraction decomposition for each rational function. Crucially, both of the quadratic polynomials s^2+1 and s^2+4 are irreducible since their roots are complex conjugates. Consequently, the form of the partial fraction decomposition is given for some real numbers A, B, C, and D by

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \text{ so that}$$

$$1 = (As+B)(s^2+4) + (Cs+D)(s^2+1) \text{ and}$$

$$1 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D).$$

Comparing the coefficients of each of the polynomials, we obtain the following system of equations.

$$A + C = 0 \qquad B + D = 0$$

$$4A + C = 0 \qquad 4B + D = 1$$

Each of the linear equations in A and C can be easily solved to find that $A = C = 0$. Likewise, we may subtract $B + D = 0$ from $4B + D = 1$ to determine that $3B = 1$ so that $B = \frac{1}{3}$ and $D = -\frac{1}{3}$.

$$\mathcal{L}\{y\} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) + \frac{1}{3}e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

Each of the rational functions with a coefficient of $e^{-\pi s}$ is the Laplace transform of $u_\pi(t)f(t-\pi)$ for some real univariate function $f(t)$ by the Scaling Property of the Laplace Transform. Bearing in mind the form of the denominator $s^2 + a^2$ for some real number a, we recognize that

$$y(t) = \frac{1}{3}\left[\sin(t) - \frac{1}{2}\sin(2t)\right] + \frac{1}{3}u_\pi(t)\left[\sin(t-\pi) - \frac{1}{2}\sin(2t-2\pi)\right]. \qquad \diamond$$

Example 3.3.7. Use the Laplace transform to solve the initial value problem.

$$y^{(4)} + 5y'' + 4y = 1 - u_\pi(t) \text{ with } y(0) = 0,\ y'(0) = 0,\ y''(0) = 0, \text{ and } y'''(0) = 0$$

Solution. We proceed by appealing to the linearity of the Laplace transform and the Recursive Formula for the Laplace Transform of a Derivative to evaluate the initial value problem as follows.

$$\mathcal{L}\{y^{(4)} + 5y'' + 4y\} = \mathcal{L}\{1 - u_\pi(t)\}$$

$$\mathcal{L}\{y^{(4)}\} + 5\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{u_\pi(t)\}$$

$$(s^4 + 5s^2 + 4)\mathcal{L}\{y\} = \frac{1}{s} - \frac{1}{s}e^{-\pi s}$$

$$\mathcal{L}\{y\} = \frac{1 - e^{-\pi s}}{s(s^4 + 5s^2 + 4)}$$

Considering that $s^4 + 5s^2 + 4 = (s^2 + 1)(s^2 + 4)$ is a product of irreducible quadratic polynomials, the partial fraction decomposition is given for some real numbers A, B, C, D, and E by

$$\frac{1}{s(s^2+1)(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{s^2+4} \text{ so that}$$

$$1 = A(s^4 + 5s^2 + 4) + (Bs + C)(s^3 + 4s) + (Ds + E)(s^3 + s) \text{ or}$$

$$1 = (A + B + D)s^4 + (C + E)s^3 + (5A + 4B + D)s^2 + (4C + E)s + 4A.$$

Consequently, we have that $4A = 1$ if $s = 0$ so that $A = \frac{1}{4}$. Comparing the coefficients of the quartic polynomial on the right with the constant polynomial on the left, we obtain a system of equations.

$$B + D + \frac{1}{4} = 0 \qquad C + E = 0$$

$$4B + D + \frac{5}{4} = 0 \qquad 4C + E = 0$$

We conclude that $3B = -1$ and $3C = 0$ so that $B = -\frac{1}{3}$, $D = \frac{1}{12}$, and $C = E = 0$. Considering this partial fraction decomposition in the context of the Laplace transform, it follows that

$$\mathcal{L}\{y\} = (1 - e^{-\pi s})\left(\frac{1}{4} \cdot \frac{1}{s} - \frac{1}{3} \cdot \frac{s}{s^2+1} + \frac{1}{12} \cdot \frac{s}{s^2+4}\right).$$

Each of the rational functions with a coefficient of $e^{-\pi s}$ is the Laplace transform of $u_\pi(t)f(t - \pi)$ for some real univariate function $f(t)$ by the Scaling Property of the Laplace Transform. Bearing in mind the form of the denominator $s^2 + a^2$ for some real number a, we conclude that

$$y(t) = \frac{1}{4} - \frac{1}{3}\cos(t) + \frac{1}{12}\cos(2t) - u_\pi(t)\left[\frac{1}{4} - \frac{1}{3}\cos(t - \pi) + \frac{1}{12}\cos(2t - 2\pi)\right]. \qquad \diamond$$

3.4 Impulse Functions

Often, in examples arising in physics and electrical engineering, it is necessary to consider physical phenomena that involve a very large force acting for a very short period of time. Concretely, we will assume that $g(t)$ is a real univariate function defined for all real numbers $t \geq 0$ with the additional property that $g(t) \gg 0$ for all real numbers $t_0 - \varepsilon < t < t_0 + \varepsilon$ for some real number t_0 and some "sufficiently small" real number $\varepsilon > 0$ and $g(t) = 0$ for all real numbers $t \leq t_0 - \varepsilon$ and $t \geq t_0 + \varepsilon$. Considering that the definite integral of $g(t)$ over the entire real line measures the area bound by the curve $g(t)$ over its domain, if $g(t)$ measures the force in a mechanical system, it follows that

$$I(\varepsilon) = \int_{-\infty}^{\infty} g(t)\,dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} g(t)\,dt$$

measures the **impulse** of the force $g(t)$ over the interval of time $(t_0 - \varepsilon, t_0 + \varepsilon)$. Perhaps the most immediate physical example relates the current $g(t)$ of an electrical circuit to its total voltage $I(\varepsilon)$.

We will find it most fruitful to henceforth direct our attention to a particularly interesting family of force functions defined for any "sufficiently small" real number $\varepsilon > 0$ by the piecewise function

$$d_\varepsilon(t) = \begin{cases} \dfrac{1}{2\varepsilon} & \text{if } -\varepsilon < t < \varepsilon \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Considering that ε is sufficiently small, it is natural to wonder what becomes of the impulse function $I(\varepsilon)$ corresponding to $d_\varepsilon(t)$ as ε tends to zero. By definition of $I(\varepsilon)$, we have that

$$I(\varepsilon) = \int_{-\infty}^{\infty} d_\varepsilon(t)\, dt = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\, dt = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} dt = 1$$

for all real numbers $\varepsilon > 0$. Consequently, the impulse function $I(\varepsilon)$ corresponding to the family of force functions $d_\varepsilon(t)$ are all identically equal to 1; on the other hand, we have that

$$\lim_{\varepsilon \to 0} d_\varepsilon(t) = 0$$

for any nonzero real number t. Explicitly, as ε tends to zero, the interval on which $d_\varepsilon(t)$ is nonzero shrinks to the origin, hence everywhere except at the origin, it follows that $d_\varepsilon(t) = 0$.

Unfortunately, at the origin, the result of the limit of $d_\varepsilon(t)$ as ε tends to zero is not well-defined; however, if we expand our view to consider **distributions** in addition to the functions with which we are presently familiar, then we obtain a meaning definition of the **unit impulse function** that is zero everywhere except at the origin and yet induces an impulse of unit magnitude at the origin. Explicitly, the **Dirac delta function** $\delta(t)$ is a distribution defined for all real numbers t satisfying

$$\delta(t) = 0 \text{ for all real numbers } t \neq 0 \text{ and } \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Considering that the Dirac delta function $\delta(t)$ is not in fact a function, we cannot define its Laplace transform according to the usual definition; rather, its Laplace transform is given as the limit

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\varepsilon \to 0} \mathcal{L}\{d_\varepsilon(t - t_0)\}.$$

On the bright side, computing this limit is feasible: if $t_0 > 0$, then as ε tends to zero, we must have that $\varepsilon < t_0$ so that $t_0 - \varepsilon > 0$. Consequently, it follows that $d_\varepsilon(t - t_0)$ is defined for $t > 0$ in the limit. Even more, this function is nonzero if and only if $t_0 - \varepsilon < t < t_0 + \varepsilon$, hence we have that

$$\lim_{\varepsilon \to 0} \mathcal{L}\{d_\varepsilon(t - t_0)\} = \lim_{\varepsilon \to 0} \int_0^{\infty} e^{-st} d_\varepsilon(t - t_0)\, dt \qquad \text{(definition of the Laplace transform)}$$

$$= \lim_{\varepsilon \to 0} \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \frac{1}{2\varepsilon} e^{-st}\, dt \qquad \text{(Refer to the above exposition.)}$$

$$= \lim_{\varepsilon \to 0} \frac{e^{-(t_0 - \varepsilon)s} - e^{-(t_0 + \varepsilon)s}}{2s\varepsilon} \qquad \text{(definition of the definite integral)}$$

$$= e^{-t_0 s} \lim_{\varepsilon \to 0} \frac{e^{\varepsilon s} - e^{-\varepsilon s}}{2s\varepsilon} \qquad \text{(Our limit depends only on } \varepsilon.\text{)}$$

$$= e^{-t_0 s} \lim_{\varepsilon \to 0} \frac{se^{\varepsilon s} + se^{-\varepsilon s}}{2s} \qquad \text{(Employ L'Hôpital's Rule.)}$$

$$= e^{-t_0 s}. \qquad \text{(Plug and chug with } \varepsilon = 0.\text{)}$$

Formula 3.4.1 (Laplace Transform of the Dirac Delta Function)**.** *Consider the Dirac delta function $\delta(t - t_0)$ centered at $t = t_0$ for any real number $t_0 \geq 0$. We have that*

$$\mathcal{L}\{\delta(t - t_0)\} = \begin{cases} e^{-t_0 s} & \text{if } t_0 > 0 \text{ and} \\ 1 & \text{if } t_0 = 0. \end{cases}$$

Proof. By the preceding exposition, we have that $\mathcal{L}\{\delta(t - t_0)\} = e^{-t_0 s}$ if $t_0 > 0$ so that

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0+} \mathcal{L}\{\delta(t - t_0)\} = \lim_{t_0 \to 0+} e^{-t_0 s} = 1. \qquad \square$$

Example 3.4.2. Use the Laplace transform to solve the initial value problem.

$$y'' + y = \delta(t - 1) \text{ with } y(0) = 1 \text{ and } y'(0) = 1$$

Solution. Like usual, we may evaluate the initial value problem by appealing to the linearity of the Laplace transform and the Recursive Formula for the Laplace Transform of a Derivative.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\delta(t - 1)\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = e^{-s}$$

$$(s^2 + 1)\mathcal{L}\{y\} - s - 1 = e^{-s}$$

$$\mathcal{L}\{y\} = \frac{e^{-s}}{s^2 + 1} + \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1}$$

$$y(t) = u_1(t)\sin(t - 1) + \cos(t) + \sin(t) \qquad \diamond$$

Example 3.4.3. Use the Laplace transform to solve the initial value problem.

$$y^{(4)} - y = \delta(t) \text{ with } y(0) = 0,\ y'(0) = 1,\ y''(0) = 0, \text{ and } y'''(0) = 2$$

Solution. Once again, the name of the game is to employ the linearity of the Laplace transform and the Recursive Formula for the Laplace Transform of a Derivative to solve the initial value problem.

$$\mathcal{L}\{y^{(4)} - y\} = \mathcal{L}\{\delta(t)\}$$

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = 1$$

$$(s^4 - 1)\mathcal{L}\{y\} - s^2 - 2 = 1$$

$$\mathcal{L}\{y\} = \frac{s^2 + 3}{s^4 - 1}$$

Carrying out the partial fraction decomposition with $s^4 - 1 = (s^2+1)(s+1)(s-1)$, we find that

$$\frac{s^2+3}{s^4-1} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s-1} \text{ so that}$$

$$s^2 + 3 = (As+B)(s^2-1) + C(s^2+1)(s-1) + D(s^2+1)(s+1) \text{ or}$$

$$s^2 + 3 = (A+C+D)s^3 + (B-C+D)s^2 + (-A+C+D)s + (-B-C+D).$$

We find that $-4C = 4$ if $s = 1$ and $4D = 4$ if $s = -1$ so that $C = -1$ and $D = 1$. Comparing the coefficients of the cubic polynomial on the right with the quadratic polynomial on the left above, we conclude that $A = 0$ and $B + 2 = 1$ so that $B = -1$. Ultimately, this yields that

$$\mathcal{L}\{y\} = -\frac{1}{s^2+1} - \frac{1}{s+1} + \frac{1}{s-1}.$$

Each of these rational functions admits a familiar inverse Laplace transform, from which we obtain

$$y(t) = -\sin(t) - e^{-t} + e^t. \qquad \diamond$$

Example 3.4.4. Use the Laplace transform to solve the initial value problem.

$$4y'' - 4y' - 3y = \delta(t-\pi) + u_{2\pi}(t) \text{ with } y(0) = 1 \text{ and } y'(0) = 0$$

Solution. We proceed as is routine by using the linearity of the Laplace transform and the Recursive Formula for the Laplace Transform of a Derivative to evaluate the initial value problem as follows.

$$\mathcal{L}\{4y'' - 4y' - 3y\} = \mathcal{L}\{\delta(t-\pi) + u_{2\pi}(t)\}$$

$$4\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\} + \mathcal{L}\{u_{2\pi}(t)\}$$

$$(4s^2 - 4s - 3)\mathcal{L}\{y\} - 4s + 4 = e^{-\pi s} + \frac{1}{s}e^{-2\pi s}$$

$$\mathcal{L}\{y\} = \frac{e^{-\pi s}}{4s^2 - 4s - 3} + \frac{e^{-2\pi s}}{s(4s^2-4s-3)} + \frac{4s-4}{4s^2-4s-3}$$

Each of these rational functions requires partial fraction decomposition so that we can more easily recognize the inverse Laplace transform. Considering that $4s^2 - 4s - 3 = (2s+1)(2s-3)$, the form of the partial fraction decomposition for each of the three cases can be determined as follows.

$$\frac{1}{4s^2 - 4s - 3} = \frac{A}{2s+1} + \frac{B}{2s-3}$$

$$1 = A(2s-3) + B(2s+1)$$

$$\frac{1}{s(4s^2 - 4s - 3)} = \frac{C}{s} + \frac{D}{2s+1} + \frac{E}{2s-3}$$

$$1 = C(2s+1)(2s-3) + Ds(2s-3) + Es(2s+1)$$

$$\frac{4s-4}{4s^2 - 4s - 3} = \frac{F}{2s+1} + \frac{G}{2s-3}$$

$$4s - 4 = F(2s-3) + G(2s+1)$$

Conveniently, the coefficients of each polynomial equation can be obtained by plugging in the roots of the linear polynomials s, $2s+1$, and $2s-3$. Carrying this out, we find that $A = -\frac{1}{4}$, $B = \frac{1}{4}$, $C = -\frac{1}{3}$, $D = \frac{1}{2}$, $E = \frac{1}{6}$, $F = \frac{3}{2}$, and $G = \frac{1}{2}$ so that the Laplace transform of $y(t)$ is

$$\mathcal{L}\{y\} = \frac{1}{4}e^{-\pi s}\left(\frac{1}{2s-3} - \frac{1}{2s+1}\right) - \frac{1}{6}e^{-2\pi s}\left(\frac{2}{s} - \frac{1}{2s-3} - \frac{3}{2s+1}\right) + \frac{1}{2}\left(\frac{1}{2s-3} + \frac{3}{2s+1}\right).$$

Considering that $2s - 3 = \frac{1}{2}\left(s - \frac{3}{2}\right)$ and $2s + 1 = \frac{1}{2}\left(s + \frac{1}{2}\right)$, we obtain the inverse Laplace transform

$$y(t) = \frac{1}{8}u_\pi(t)(e^{3(t-\pi)/2} - e^{-(t-\pi)/2}) - \frac{1}{12}u_{2\pi}(t)(4 - e^{3(t-2\pi)/2} - 3e^{-(t-2\pi)/2}) + \frac{1}{4}(e^{3t/2} + 3e^{-t/2}). \quad \diamond$$

We summarize our discussion of the Laplace transform and its usefulness in the solution of initial value problems with nth order linear ordinary differential equations in the following algorithm.

Algorithm 3.4.5 (Laplace Transform Solutions of Initial Value Problems with Constant Coefficients)**.** Given any integer $n \geq 1$, any real univariate functions $y(t)$ and $g(t)$ defined for all real numbers $t \geq 0$, and any real numbers $a_1, \dots, a_{n-1}, a_n, t_0, \dots, t_{n-1}$, consider the initial value problem

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = g(t) \text{ with } y^{(i)}(0) = t_i \text{ for each integer } 0 \leq i \leq n-1.$$

Carry out the following steps to determine a solution $y = f(t)$ for all real numbers $t \geq 0$.

1.) Compute the Laplace transform $\mathcal{L}\{g(t)\}$ of $g(t)$ using the linearity of the Laplace transform.

2.) Compute the Laplace transform of $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y$ using the linearity of the Laplace transform and the Recursive Formula for the Laplace Transform of a Derivative to express $\mathcal{L}\{y^{(i)}\}$ as a real polynomial in s of degree i with leading coefficient $\mathcal{L}\{y\}$.

3.) By simplifying the equation involving $\mathcal{L}\{y\}$, it ought to be possible to obtain an equation of the form $(s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n)\mathcal{L}\{y\} = h(s)$ for some real univariate function $h(s)$.

4.) By factoring this polynomial in s into linear and irreducible quadratic polynomials, the method of partial fraction decomposition yields an expression of $\mathcal{L}\{y\}$ in terms of rational functions.

5.) Compute the inverse Laplace transform of each rational function to determine $y = f(t)$.

We conclude this section with a table of Laplace transforms we have derived in these notes.

Table 3.4.6 (Common Laplace Transforms)**.** Consider any real numbers a, $c \geq 0$, and $t_0 \geq 0$.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	domain of $F(s)$	derivation in the notes
a	$\frac{a}{s}$	$s > 0$	Example 3.1.5
e^{at}	$\frac{1}{s-a}$	$s > a$	Example 3.1.6
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$	Example 3.1.7
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$	Quiz 14 Solutions
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$	Example 3.1.11
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$	$s > a$	Corollary 3.2.2
$e^{at} f(t)$	$F(s-a)$	$s > a$	Proposition 3.2.5
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$	Use the table.
$u_c(t)$	$\frac{1}{s} e^{-cs}$	$s > 0$	Equation 3.3.1
$u_c(t) f(t-c)$	$e^{-cs} F(s)$	$s > a$	Proposition 3.3.4
$\delta(t - t_0)$	$e^{-t_0 s}$	$s > 0$	Formula 3.4.1
$(f * g)(t)$	$F(s)G(s)$	$s > a$	Theorem 3.5.5

3.5 Convolution and the Convolution Integral

We have thus far in this chapter developed a general theory to solve initial value problems involving linear ordinary differential equations with constant coefficients via the Laplace transform. Explicitly, the linearity of the Laplace transform allows us to express the Laplace transform of the function

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

as the difference of $(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0)\mathcal{L}\{y\}$ and a real polynomial in s. Unfortunately, we are not able to employ this technique to solve initial value problems involving ordinary differential equations with coefficients that are not constant because the Laplace transform does not commute with multiplication of functions. Concretely, according to Table 3.4.6, we have that

$$\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2} \neq \frac{1}{s^2(s-1)} = \mathcal{L}\{t\}\mathcal{L}\{e^t\},$$

hence the Laplace transform of te^t is not the product of the Laplace transform of t and the Laplace transform of e^t. Countless examples along these lines illustrate this failure. Luckily, we may define a "generalized" product of continuous functions over which the Laplace transform commutes.

Definition 3.5.1. Given any real univariate functions $f(t)$ and $g(t)$ that are continuous for all real numbers $t \geq 0$ (except possibly at countably infinitely many non-negative real numbers), the **convolution** of $f(t)$ and $g(t)$ is the real univariate function defined by the **convolution integral**

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,dx.$$

Even though we will not endeavor to prove the following proposition in the notes, we encourage the interested reader to verify the following properties for their own edification and practice.

Proposition 3.5.2 (Properties of Convolution)**.** *Given any real univariate functions $f(t)$ and $g(t)$ that are continuous for all real numbers $t \geq 0$ (except possibly at countably infinitely many non-negative real numbers), the following properties hold for the convolution $(f * g)(t)$ of $f(t)$ and $g(t)$.*

1.) (Associativity) *We have that $[f * (g * h)](t) = [(f * g) * h](t)$ for all real numbers $t \geq 0$.*

2.) (Commutativity) *We have that $(f * g)(t) = (g * f)(t)$ for all real numbers $t \geq 0$.*

3.) (Distributivity) *We have that $[f * (g + h)](t) = (f * g)(t) + (f * h)(t)$ for all real numbers $t \geq 0$.*

4.) (Homogeneity) *We have that $(f * 0)(t) = 0$ for all real numbers t.*

Example 3.5.3. Compute the convolution $(f * 1)(t)$ of the function $f(t) = \cos(t)$.

Solution. By definition, the convolution of $\cos(t)$ and 1 is given by

$$(f * 1)(t) = \int_0^t \cos(t-x)\,dx = [-\sin(t-x)]_0^t = \sin(t).$$

Consequently, it is not necessarily the case that $(f * 1)(t) = f(t)$; however, one could demonstrate (with a bit of thought) that for the Dirac delta function $\delta(t)$, it holds that $(f * \delta)(t) = f(t)$. ⋄

Example 3.5.4. Compute the convolution $(f * f)(t)$ of the function $f(t) = \sin(t)$.

Solution. By definition, the convolution of $\sin(t)$ with itself is given by

$$\begin{aligned}(f * f)(t) &= \int_0^t \sin(t - x)\sin(x)\,dx \\ &= \int_0^t [\sin(t)\cos(x) - \sin(x)\cos(t)]\sin(x)\,dx && \text{(Double Angle Formula)} \\ &= \sin(t)\int_0^t \sin(x)\cos(x)\,dx - \cos(t)\int_0^t \sin^2(x)\,dx \\ &= \sin(t)\left[\frac{1}{2}\sin^2(x)\right]_0^t - \cos(t)\int_0^t \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]dx && \text{(Power Reduction Formula)} \\ &= \frac{1}{2}\sin^3(t) - \frac{t}{2}\cos(t) + \frac{1}{4}\sin(2t)\cos(t).\end{aligned}$$

Observe that $\sin\left(\frac{3\pi}{2}\right) = -1$ and $\cos\left(\frac{3\pi}{2}\right) = 0$ so that $(f * f)\left(\frac{3\pi}{2} + 2k\pi\right) = -\frac{1}{2}$ for any integer k. Consequently, it is not necessarily the case that $(f * f)(t)$ is non-negative. ◇

We turn our attention at last to the usefulness of the convolution integral with respect to the Laplace transform. Like we indicated at the beginning of the section, the Laplace transform enjoys the commutative property with respect to convolution, according to the following principal theorem.

Theorem 3.5.5 (Commutative Law for the Laplace Transform and Convolution)**.** *Given any real univariate functions $f(t)$ and $g(t)$ with respective Laplace transforms $F(s)$ and $G(s)$ defined for all real numbers $s > a$ for some real number $a \geq 0$, the Laplace transform of $(f * g)(t)$ satisfies that*

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s) \text{ for all real numbers } s > a.$$

Proof. We will compute the product $F(s)G(s)$ and verify that it is indeed the Laplace transform of $(f * g)(t)$. By definition of the Laplace transform of $f(t)$ and $g(t)$, we have that

$$\begin{aligned}F(s)G(s) &= \left(\int_0^\infty e^{-sx} f(x)\,dx\right)\left(\int_0^\infty e^{-sy} g(y)\,dy\right) \\ &= \int_0^\infty \left(\int_0^\infty e^{-sy} g(y)\,dy\right) e^{-sx} f(x)\,dx \\ &= \int_0^\infty \left(\int_0^\infty e^{-s(x+y)} f(x)g(y)\,dy\right) dx.\end{aligned}$$

Explicitly, the improper integral in x does not depend on y, so we may commute the integral in x with the terms involving y. Likewise, the improper integral in y does not depend on x, so we may commute the integral in y with the terms involving x. We are now in a position to make an inspired substitution $u = x + y$ with $du = dy$. Observe that for $y = 0$, we have that $u = x$ and u tends to infinity as y tends to infinity. By solving for y, we find that $y = u - x$ so that

$$F(s)G(s) = \int_0^\infty \left(\int_0^\infty e^{-s(x+y)} f(x)g(y)\,dy \right) dx$$

$$= \int_0^\infty \left(\int_x^\infty e^{-su} f(x)g(u-x)\,du \right) dx.$$

Crucially, we note that the region of integration is defined for all real numbers $u \geq x \geq 0$. Granted that Fubini's Theorem applies (so that we may interchange the order of integration), we obtain

$$F(s)G(s) = \int_0^\infty \left(\int_x^\infty e^{-su} f(x)g(u-x)\,du \right) dx$$

$$= \int_0^\infty \int_x^\infty e^{-su} f(x)g(u-x)\,du\,dx$$

$$= \int_0^\infty \int_0^u e^{-su} f(x)g(u-x)\,dx\,du$$

$$= \int_0^\infty e^{-su} \left(\int_0^u f(x)g(u-x)\,dx \right) du$$

$$= \int_0^\infty e^{-su} (f * g)(u)\,du$$

$$= \mathcal{L}\{(f * g)(t)\}.$$

Last, we note that $\mathcal{L}\{(f * g)(t)\}$ is defined so long as $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ are defined. By hypothesis, we conclude that $\mathcal{L}\{(f * g)(t)\}$ is defined for all real numbers $s > a$. □

Example 3.5.6. Compute the inverse Laplace transform of $F(s) = \dfrac{1}{s^3 - s^2}$.

Solution. By the Commutative Law for the Laplace Transform and Convolution, it suffices to view $F(s)$ as a product $G(s)H(s)$ of Laplace transforms of real univariate functions $g(t)$ and $h(t)$.

$$F(s) = \frac{1}{s^3 - s^2} = \frac{1}{s^2(s-1)} = \frac{1}{s^2} \cdot \frac{1}{s-1} = G(s)H(s)$$

Crucially, we note that $G(s) = \mathcal{L}\{t\}$ and $H(s) = \mathcal{L}\{e^t\}$ so that $g(t) = t$ and $h(t) = e^t$. We are now in a position to determine the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\} = (g * h)(t)$ as follows.

$$(g * h)(t) = \int_0^t (t - x)e^x\,dx$$

$$= t \int_0^t e^x \, dx - \int_0^t xe^x \, dx \qquad \text{(Expand and simplify.)}$$

$$= t(e^t - 1) - \left([xe^x]_0^t - \int_0^t e^x \, dx \right) \qquad (u = x \text{ and } dv = e^x \, dx)$$

$$= t(e^t - 1) - \left(te^t - e^t + 1 \right)$$

$$= e^t - t - 1$$

Consequently, we find that $\mathcal{L}^{-1}\{F(s)\} = f(t) = e^t - t - 1$ for all real numbers $t \geq 0$. ◇

Example 3.5.7. Compute the inverse Laplace transform of $F(s) = \dfrac{2s}{(s^2+4)^2}$.

Solution. By the Commutative Law for the Laplace Transform and Convolution, it suffices to view $F(s)$ as the product $G(s)H(s)$ of Laplace transforms of real univariate functions $g(t)$ and $h(t)$.

$$F(s) = \frac{2s}{(s^2+4)^2} = \frac{2}{s^2+4} \cdot \frac{s}{s^2+4} = G(s)H(s)$$

Crucially, we have that $G(s) = \mathcal{L}\{\sin(2t)\}$, $H(s) = \mathcal{L}\{\cos(2t)\}$, $g(t) = \sin(2t)$, and $h(t) = \cos(2t)$. We are now in a position to determine the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\} = (g * h)(t)$.

$$(g * h)(t) = \int_0^t \sin(2t - 2x)\cos(2x) \, dx$$

$$= \int_0^t [\sin(2t)\cos(2x) - \sin(2x)\cos(2t)] \cos(2x) \, dx \qquad \text{(Difference Formula for Sine)}$$

$$= \sin(2t) \int_0^t \cos^2(2x) \, dx - \cos(2t) \int_0^t \sin(2x)\cos(2x) \, dx \qquad \text{(Expand and simplify.)}$$

$$= \sin(2t) \int_0^t \left[\frac{1}{2} + \frac{1}{2}\cos(4x) \right] dx - \cos(2t) \int_0^{\sin(2t)} \frac{1}{2} u \, du \qquad \text{(Double Angle Formula)}$$

$$= \sin(2t) \left[\frac{x}{2} + \frac{1}{8}\sin(4x) \right]_0^t - \frac{1}{4}\sin^2(2t)\cos(2t)$$

$$= \sin(2t) \left[\frac{t}{2} + \frac{1}{8}\sin(4t) \right] - \frac{1}{4}\sin^2(2t)\cos(2t)$$

Consequently, we find that $\mathcal{L}^{-1}\{F(s)\}$ is given as above for all real numbers $t \geq 0$. ◇

We conclude this section with a discussion of the importance of the convolution integral in the study of linear ordinary differential equations with constant coefficients. Given any real numbers a, b, and c and any real univariate function $g(t)$, consider the linear ordinary differential equation

$$ay'' + by' + cy = g(t) \text{ with } y(0) = y_0 \text{ and } y'(0) = y_0' \tag{3.5.1}$$

for some real numbers y_0 and y_0'. Like usual, the linearity of the Laplace transform and the Recursive Formula for the Laplace Transform of a Derivative yield a subsequent equation

$$(as^2 + bs + c)\mathcal{L}\{y\} - (as + b)y_0 - ay_0' = G(s) \tag{3.5.2}$$

for the Laplace transform $G(s) = \mathcal{L}\{g(t)\}$. By solving for $\mathcal{L}\{y\}$, we obtain a third equation

$$\mathcal{L}\{y\} = \frac{(as + b)y_0 + ay_0'}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}. \tag{3.5.3}$$

Conveniently, if we adopt some notation for each of the functions on the right-hand side

$$\Phi(s) = \frac{(as + b)y_0 + ay_0'}{as^2 + bs + c} \text{ and } \Psi(s) = \frac{G(s)}{as^2 + bs + c}, \tag{3.5.4}$$

then by the linearity of the inverse Laplace transform, we obtain the solution $y(t) = \phi(t) + \psi(t)$ for the inverse Laplace transforms $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ and $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$. Even more, it is crucial to note that $\phi(t)$ is the solution of the corresponding homogeneous initial value problem

$$ay'' + by' + cy = 0 \text{ with } y(0) = y_0 \text{ and } y'(0) = y_0'$$

and $\psi(t)$ is the solution of the corresponding non-homogeneous initial value problem

$$ay'' + by' + cy = g(t) \text{ with } y(0) = 0 \text{ and } y'(0) = 0.$$

By the method of partial fraction decomposition, it is possible to express $\Phi(s)$ as a sum of rational functions whose inverse Laplace transforms can be determined by referring to the table of Common Laplace Transforms. On the other hand, it is possible to perform partial fraction decomposition to express $\Psi(s)$ as a product of $G(s)$ and a sum of rational functions; then, by the Commutative Law for the Laplace Transform and Convolution, we may at last write $\Psi(s)$ as a convolution integral. Explicitly, if we denote by $H(s) = (as^2 + bs + c)^{-1}$ the **transfer function** of Equation 3.5.1, then

$$\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\} = \mathcal{L}^{-1}\{G(s)H(s)\} = (g * h)(t) = \int_0^t g(t - x)h(x)\,dx. \tag{3.5.5}$$

Example 3.5.8. Give the solution of the initial value problem in terms of a convolution integral.

$$y'' + 2y' - 15y = g(t) \text{ with } y(0) = 0 \text{ and } y'(0) = 0$$

Solution. By the above exposition, we recognize that $\Phi(s) = 0$ and $\Psi(s) = G(s)H(s)$ for the transfer function $H(s) = (s^2 + 2s - 15)^{-1}$. Consequently, in terms of a convolution integral, the solution is

$$y = \psi(t) = \mathcal{L}^{-1}\{\Psi(s)\} = (g * h)(t) = \int_0^t g(t - x)h(x)\,dx \text{ with } h(t) = \mathcal{L}^{-1}\{H(s)\}.$$

We may employ the method of partial fraction decomposition to express $H(s)$ as a sum of rational functions whose inverse Laplace transforms can be easily deduced. Explicitly, we have that

$$H(s) = \frac{1}{s^2 + 2s - 15} = \frac{1}{(s-3)(s+5)} = \frac{A}{s-3} + \frac{B}{s+5}$$

for some real numbers A and B. Unravelling this equation yields that

$$1 = A(s+5) + B(s-3)$$

so that $A = \frac{1}{8}$ for $s = 3$ and $B = -\frac{1}{8}$ for $s = 5$. We conclude that $H(s) = \frac{1}{8}(s-3)^{-1} - \frac{1}{8}(s+5)^{-1}$ and $h(t) = \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t}$. Bearing this in mind, we obtain the solution as a convolution integral

$$y = \psi(t) = \frac{1}{8}\int_0^t \big(e^{3x} - e^{-5x}\big) g(t-x)\, dx.$$

Crucially, for many cases of the univariate real function $g(t)$, it ought to be possible to compute the convolution integral to determine $y = \psi(t)$ using power series or numerical integration. ⋄

Example 3.5.9. Give the solution of the initial value problem in terms of a convolution integral.

$$y^{(4)} - 2y'' + y = g(t) \text{ with } y(0) = 0,\ y'(0) = 0,\ y''(0) = 0, \text{ and } y'''(0) = 0$$

Solution. We repeat the process outlined in the above exposition to find that

$$\mathcal{L}\{y\} = \frac{G(s)}{s^4 - 2s^2 + 1} = \frac{G(s)}{(s^2-1)^2} = \frac{G(s)}{(s-1)^2(s+1)^2}.$$

Consequently, we must employ the method of partial fraction decomposition to obtain an expression of $\mathcal{L}\{y\}$ in terms of some rational functions whose denominators we recognize in terms of the inverse Laplace transform. Explicitly, we seek real numbers A, B, C, and D such that

$$\frac{1}{(s-1)^2(s+1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} \text{ or}$$

$$1 = A(s-1)(s+1)^2 + B(s+1)^2 + C(s-1)^2(s+1) + D(s-1)^2.$$

We note that it is possible to obtain $B = \frac{1}{4}$ and $D = \frac{1}{4}$ by plugging in $s = 1$ and $s = -1$, respectively. Comparing coefficients after simplifying this cubic polynomial in s, we find that

$$1 = (A+C)s^3 + \left(A - C + \frac{1}{2}\right)s^2 + (-A-C)s + \left(-A + C + \frac{1}{2}\right)$$

so that $A + C = 0$ and $A - C = -\frac{1}{2}$. We conclude that $A = -\frac{1}{4}$ and $C = \frac{1}{4}$ so that

$$\mathcal{L}\{y\} = -\frac{1}{4}\cdot\frac{G(s)}{s-1} + \frac{1}{4}\cdot\frac{G(s)}{(s-1)^2} + \frac{1}{4}\cdot\frac{G(s)}{s+1} + \frac{1}{4}\cdot\frac{G(s)}{(s+1)^2}.$$

By the table of Common Laplace Transforms, we can readily determine that $(s-1)^{-1} = \mathcal{L}^{-1}\{e^t\}$, $(s-1)^{-2} = \mathcal{L}^{-1}\{te^t\}$, $(s+1)^{-1} = \mathcal{L}^{-1}\{e^{-t}\}$, and $(s+1)^{-2} = \mathcal{L}^{-1}\{te^{-t}\}$, hence we conclude that

$$y = \psi(t) = \frac{1}{4}\int_0^t \big(-e^x + xe^x + e^{-x} + xe^{-x}\big) g(t-x)\, dx.$$

Even though the usefulness of this expression may not be immediate to the reader, we point out that in many cases, it is possible to evaluate the convolution integral by a power series or with numerical integration. Concretely, for many applications, it is not necessary to compute the integral explicitly; we only need to determine an approximation of $y = \psi(t)$ to some satisfactory degree of accuracy. ⋄

3.6 Euler and Backward Method

We have at this point exhausted all of the analytical techniques used to solve ordinary differential equations that we will cover in this course. Briefly summarized, these methods include

- the Leibniz Formula for Solutions of First Order Linear Ordinary Differential Equations,
- simplification and direct integration of Separable First Order Equations,
- Constructing an Exact Integrating Factor to obtain Exact First Order Equations,
- construction and use of the characteristic equation according to the Solutions of Homogeneous Second Order Linear Ordinary Differential Equation with Constant Coefficients III algorithm,
- construction and use of the characteristic equation in tandem with The Method of Undetermined Coefficients and the Variation of Parameters to solve non-homogeneous second order linear ordinary differential equations with constant coefficients,
- determination of a recurrence relation to obtain Power Series Solutions of Linear Equations for second order linear equations with non-constant coefficients,
- construction and use of Euler's characteristic equation for Solutions of Euler's Equation, and
- the Recursive Formula for the Laplace Transform of a Derivative for initial value problems.

We turn our attention next to numerical methods of solving ordinary differential equations. Unfortunately, in many of the differential equations that arise in applications, these analytical techniques either do not apply (e.g., if the equation is non-linear) or present computational difficulties (e.g., if the equation involves more complicated functions than we have discussed). Bearing this in mind, we seek techniques that allow us to accurately approximate solutions of ordinary differential equations without any requirement of writing down a closed-form expression for the solution.

We will for the sake of simplicity focus primarily on the first order initial value problem

$$\frac{dy}{dt} = f(t, y) \text{ with } y(t_0) = y_0 \tag{3.6.1}$$

for some real numbers t_0 and y_0. Even more, we will assume that the real bivariate functions f and f_y are continuous on some open rectangle in the ty-plane that contains the point (t_0, y_0). By the Fundamental Theorem of First Order Ordinary Differential Equations, there exists an open interval that contains t_0 for which there exists a unique solution $y = \phi(t)$ of Equation 3.6.1.

Definition 3.6.1. Given any real bivariate function $f(t, y)$ such that f and f_y are continuous on some open rectangle in the ty-plane that contains the point (t_0, y_0), the **Euler Method** is a numerical technique that can be used to solve any first order initial value problem of the form

$$\frac{dy}{dt} = f(t, y) \text{ with } y(t_0) = y_0.$$

Explicitly, the Euler Method is defined recursively by the first order recurrence relation

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n) \text{ for each integer } n \geq 0. \tag{3.6.2}$$

Definition 3.6.2. Given any real bivariate function $f(t, y)$ such that f and f_y are continuous on some open rectangle in the ty-plane that contains the point (t_0, y_0), the **Backward Euler Method** is a numerical technique that can be used to solve any first order initial value problem of the form

$$\frac{dy}{dt} = f(t, y) \text{ with } y(t_0) = y_0.$$

Explicitly, the Backward Euler Method is defined recursively by the first order recurrence relation

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})(t_{n+1} - t_n) \text{ for each integer } n \geq 0. \tag{3.6.3}$$

Crucially, we note that this formula determines the sequence y_{n+1} implicitly in terms of t_n and y_n as opposed to the Euler Method that provides an explicit formula for y_{n+1} in terms of t_n and y_n, hence the difficulty in carrying out the Backward Euler Method largely depends on $f(t, y)$.

Often, it is most convenient in practice to carry out the Euler Method and the Backward Euler Method with a fixed **step size** $h = t_{n+1} - t_n$ for all integers $n \geq 0$, as in the following examples.

Example 3.6.3. Compare the Euler and Backward Euler Method to solve the initial value problem

$$y' = t^2 - 1 + 2y \text{ with } y(0) = -1$$

on the closed interval $0 \leq t \leq 1$ with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$.

Example 3.6.4. Compare the Euler and Backward Euler Method to solve the initial value problem

$$y' = \frac{y^2}{t^2 + 1} \text{ with } y(0) = 1$$

on the closed interval $0 \leq t \leq 1$ with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$.

We note that there are many ways to devise numerical methods. Often, these techniques are constructed by estimating values defined by limits in precise contexts. Explicitly, the Euler Method and the Backward Method are both derived from the limit definition of the derivative

$$y'(t) = \lim_{t_{n+1} \to t_n} \frac{y(t_{n+1}) - y(t_n)}{t_{n+1} - t_n}$$

by assuming that $h = t_{n+1} - t_n$ is "sufficiently small." Once a numerical method has been defined, its **convergence** is a critical concern. Concretely, we seek to determine the accuracy of a solution obtained numerical as the step size $h = t_{n+1} - t_n$ tends to zero. We will assume that $y = \phi(t)$ is a solution of the underlying differential equation. We define the **global truncation error**

$$E_n = \phi(t_n) - y_n$$

as the sequence of differences between the actual value $\phi(t_n)$ of the solution at the real number t_n and the approximate value y_n of the solution at the real number t_n for each integer $n \geq 0$. Given that the approximate value y_n and the actual value $\phi(t_n)$ coincide, the **local truncation error** in proceeding one step is due to the nature of the approximation; as such, it is more subtle.

Formula 3.6.5 (Local Truncation Error Formula for the Euler Method)**.** Given any real bivariate function $f(t, y)$ such that f, f_t, and f_y are continuous on some open rectangle in the ty-plane that contains the point (t_0, y_0), the local truncation error for the Euler Method is given by

$$e_{n+1} = \phi(t_{n+1}) - y_{n+1} = \frac{1}{2}\phi''(\tilde{t}_n)(t_{n+1} - t_n)$$

for some real number $t_n < \tilde{t}_n < t_{n+1}$ for each integer $n \geq 0$. Consequently, we have that

$$|e_n| \leq \frac{1}{2}\max\{|\phi''(t)| : a \leq t \leq b\}(t_{n+1} - t_n)$$

for the interval of definition $[a, b]$ of the unique solution $y = \phi(t)$ of the differential equation.

Example 3.6.6. Construct a formula for the local truncation error for the Euler Method if

$$y' = ty \text{ and } y(0) = 1.$$

Example 3.6.7. Construct a formula for the local truncation error for the Euler Method if

$$y' = \frac{y^2}{t^2 + 1} \text{ and } y(0) = 1.$$

3.7 Runge-Kutta Method

We point out that the Euler Method belongs to a more general class of numerical methods.

Definition 3.7.1. Given any real bivariate function $f(t, y)$ such that f and f_y are continuous on some open rectangle in the ty-plane that contains the point (t_0, y_0), the **Runge-Kutta Method** is a numerical technique that can be used to solve any first order initial value problem of the form

$$\frac{dy}{dt} = f(t, y) \text{ with } y(t_0) = y_0.$$

Explicitly, the Runge-Kutta Method is defined recursively by the first order recurrence relation

$$y_{n+1} = y_n + \frac{1}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})(t_{n+1} - t_n) \text{ for each integer } n \geq 0. \tag{3.7.1}$$

Each of the coefficients k_{ni} is defined recursively as follows for a fixed step size $h = t_{n+1} - t_n$.

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right) \\ k_{n3} &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right) \\ k_{n4} &= f(t_n + h, y_n + hk_{n3}) \end{aligned} \tag{3.7.2}$$

Example 3.7.2. Carry out the Runge-Kutta Method to solve the initial value problem

$$y' = t^2 - 1 + 2y \text{ with } y(0) = -1$$

on the closed interval $0 \leq t \leq 1$ with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$.

Example 3.7.3. Carry out the Runge-Kutta Method to solve the initial value problem

$$y' = \frac{y^2}{t^2 + 1} \text{ with } y(0) = 1$$

on the closed interval $0 \leq t \leq 1$ with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$.

References

[BD09] William E. Boyce and Richard C. DiPrima. *Elementary Differential Equations and Boundary Value Problems*. 9th ed. John Wiley & Sons, Inc., 2009.

[Bec22] Dylan C. Beck. *MA172: Calculus II*. 2022. URL: https://dylan-c-beck.github.io/ma172_MA172%20Lecture%20Notes.pdf.

[RA15] Jon Rogawski and Colin Adams. *Calculus: Early Transcendentals*. 3rd ed. W.H. Freeman, 2015.

[Ste07] James Stewart. *Essential Calculus*. Brooks/Cole, Cengage Learning, 2007.